Chapter 10

Mapping I

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In the Chapters 2 & 4, the conceptual basis and formulation of aperture synthesis in Radio Astronomy has been described. In particular, it has been shown that (1) an interferometer records the mutual coherence function, also called the visibility of the signals from the sky, and (2) the visibility is the Fourier transform of the sky brightness distribution. This chapter describes the coordinate systems used in practical aperture synthesis in Radio Astronomy and presents the derivation of the 2D Fourier transform relation between the visibility and the brightness distribution.

10.1 Coordinate Systems

10.1.1 Angular Co-ordinates

As described in Chapter 4, the response of an interferometer to quasi-monochromatic radiation from a point source located at the phase center is given by

$$r(\tau(t)) = cos(2\pi\nu_o\tau), \tag{10.1.1}$$

where $\tau = \tau_o = (D/c)sin(\theta(t))$ is the geometrical delay, θ is the direction which the antennas are tracking with respect to the vertical direction, λ is the wavelength, ν_o is the center frequency of the observing band and D is the separation between the antennas. As the antennas track the source, the geometrical delay changes as a function of time. This changing τ is exactly compensated with a computer controlled delay and for a point source at the phase center, the output of the interferometer is the amplitude of the fringe pattern.

For a source located at an angle $\theta = \theta_o+\Delta\theta$, for small $\Delta\theta$, $\tau = \tau_o+\Delta\theta(D/c)cos(\theta(t))$. Since fringe stopping compensates for τ_o, the response of the interferometer for a source $\Delta\theta$ away from the phase center is $cos(2\pi\Delta\theta D_\lambda cos(\theta))$ where $D_\lambda = D/\lambda$. If the phase center is shifted by equivalent of $\lambda/4$, the interferometer will pick up an extra phase of $\pi/2$ and the response will be sinusoidal instead of co-sinusoidal. Hence, an interferometer responds to both even and odd part of the brightness distribution. Interferometer response can then be written in complex notation as

$$r(\tau(t)) = e^{2\pi\iota\Delta\theta D_\lambda cos(\theta)}. \tag{10.1.2}$$

Writing $u = D_\lambda cos(\theta)$, which is the projected separation between the antennas in units of wavelength in the direction normal to the phase center and $l = sin(\Delta\theta) \approx \Delta\theta$, we get

$$r(u,l) = e^{2\pi\iota ul} = e^{2\pi\iota u\Delta\theta} \qquad (10.1.3)$$

as the complex response of a two element interferometer for a point source of unit flux located $\Delta\theta$ away from the phase center given by the direction θ_o.

Usually the phase center coincides with the center of the field being tracked by all the antennas. Let the normalized power reception patter of antennas (which are assumed to be identical) at a particular frequency be $B(\Delta\theta)$ and the surface brightness of an extended source be represented by $I(\Delta\theta)$. The response of the interferometer to a point source located $\Delta\theta$ away from the phase center would then be $I(\Delta\theta)B(\Delta\theta)e^{2\pi\iota u\Delta\theta}$. For an extended source with a continuous surface brightness distribution, the response is given by

$$V(u) = \int B(\Delta\theta)I(\Delta\theta)e^{2\pi\iota u\Delta\theta}d\Delta\theta = \int B(l)I(l)e^{2\pi\iota ul}dl. \qquad (10.1.4)$$

The above equation is a 1D Fourier transform relation between the source brightness distribution and the output of the visibility function V. The integral is over the entire sky visible to the antennas but is finite only for a range of l limited by the antenna primary reception pattern $B(l)$. In practice, u is calculated as a function of the source position in the sky, specified in astronomical co-ordinate system, as seen by the observer on the surface of the earth.

l in the above equation is the direction of the elemental source flux relative to the pointing center. u then has the interpretation of spatial frequency and $V(u)$ represents the 1D spatial frequency spectrum of the source.

10.1.2 Astronomical Co-ordinate System

The position of a source in the sky can be specified in various spherical coordinates systems in astronomy, differing from each other by the position of the origin and orientation of the axis. The position of the sources are specified using the azimuth and elevation angles in these coordinate systems. In the Equatorial Co-ordinate system the source position is specified by the Declination (δ) which is the elevation of the source from the normal to the celestial equator and the Right Ascension (RA), which is the azimuthal anlge from a reference position ("the first point of Aries"). The reference direction for RA is line of intersection of the equatorial and Ecliptic planes. The position of the source in the sky, in this coordinate system, remains constant as earth rotates. The azimuth and elevation of the antennas, which rotate with earth, are constantly adjusted to track a point in the sky specified by (RA, δ) coordinates. The changing position of the sources in the sky, as seen by the observer on the surface of earth is specified by replacing RA by Hour Angle (HA), which is the azimuth of the source measured in units of time, with respect to the local meridian of the source with $HA = -6^h$ pointing due East.

10.1.3 Physical Coordinate System

The antennas are located on the surface and rotate with respect to a source in the sky due the rotation of the earth. For aperture synthesis the antenna positions are specified in a co-ordinate system such that the separation of the antennas is the projected separation in plane normal to the phase center. In other words, in such a co-ordinate system the separation between the antennas is as seen by the observer sitting in the source reference frame. This system, shown in Fig 10.1, is the right-handed (u, v, w) coordinate system fixed on the surface of the earth at the array reference point, with the (u, v) plane always parallel to the tangent plane in the direction of phase center on the celestial sphere and the w axis along the direction of phase center. The u axis is along the astronomical E-W direction and v axis is along the N-S direction. The (u, v) co-ordinates of the antennas are the E-W and N-S components of position vectors. As the earth rotates, the (u, v) plane rotates with the source in the sky, changing the (u, v, w) coordinates of the antennas, generating tracks in the uv-plane.

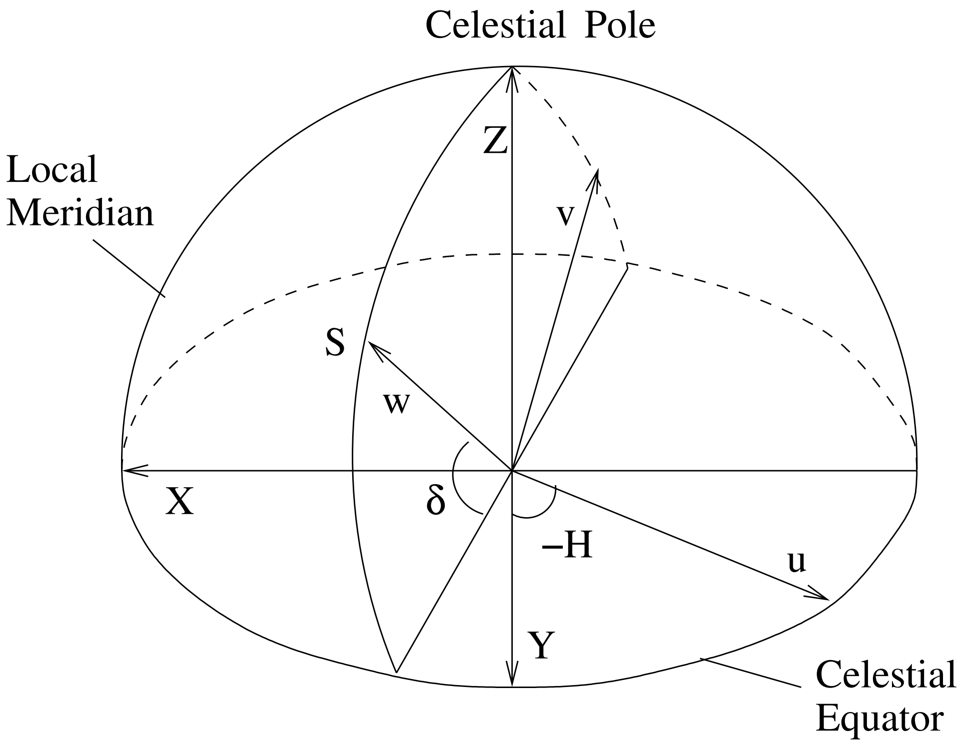

Figure 10.1: Relationship between the terrestrial co-ordinates (X,Y,Z) and the (u, v, w) co-ordinate system. The (u, v, w) system is a right handed system with the w axis pointing to the source S.

In the above formulation, the u co-ordinate of one antenna is with respect to the other antenna making the interferometer, which is located at the origin. If the origin is arbitrarily located and the co-ordinates of the two antennas are u_1 and u_2, Eq. 10.1.3 becomes

$$r(u, l) = e^{2\pi\iota(u_1-u_2)l}. \tag{10.1.5}$$

Since only the relative positions of the antennas with respect to each other enter the equations, it is only useful to work with difference between the position vectors of various antennas in the (u, v, w) co-ordinate system. The relative position vectors are called “Baseline vectors” and their lengths referred to as “baseline length”.

The source surface brightness distribution is represented as a function of the direction cosines in the (u, v, w) coordinate system. In Eq. 10.1.4 above, l is the direction cosine. The source coordinate system, which is flat only for small fields of view, is represented by (l, m, n). Since this coordinate system represents the celestial sphere, n is not an independent coordinate and is constrained to be $n = \sqrt{1 - l^2 - m^2}$.

10.1.4 Coordinate Transformation

To compute the (u, v, w) co-ordinates of the antennas, the antenna locations must first be specified in a terrestrial co-ordinate system. The terrestrial coordinate system generally used to specify the position of the antennas is a right-handed Cartesian coordinate system as shown in Figure 10.2. The (X, Y) plane is parallel to the earth's equator with X in the meridian plane and Y towards east. Z points towards the north celestial pole. In terms of the astronomical coordinate system (HA, δ), $X = (0^h, 0^o)$, $Y = (-6^h, 0^o)$ and $Z = (\delta = 90^o)$. If the components of $\bar{D}_\lambda$ are $(X_\lambda, Y_\lambda, Z_\lambda)$, then the components in the (u, v, w) system can be expressed as

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} sin(HA) & cos(HA) & 0 \\ -sin(\delta)cos(HA) & sin(\delta)sin(HA) & cos(\delta) \\ cos(\delta)cos(HA) & -cos(\delta)sin(HA) & sin(\delta) \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}. \qquad (10.1.6)$$

As earth rotates, the HA of the source changes continuously, generating different set of (u, v, w) co-ordinates for each antenna pair at each instant of time. The locus of projected antenna-spacing components u and v defines an ellipse with hour angle as the variable given by

$$u^2 + \left(\frac{v - Zcos\delta_o}{sin\delta_o}\right)^2 = X^2 + Y^2, \qquad (10.1.7)$$

where (HA_o, δ_o) defines the direction of phase center. In the uv-plane, this is an ellipse, referred to as the uv-track with HA changing along the ellipse. The pattern generated by all the uv points sampled by the entire array of antennas over the period of observation is referred to as the uv-*coverage* and as is clear from the above transformation matrix, is different for different δ. Examples of uv-coverage for a few declinations for full synthesis with GMRT array are shown in Figure 10.4.

The uv domain is the spatial frequency domain and uv-coverage represent the spatial frequencies sampled by the array. The shorter baselines (uv points closer to the origin) provide the low resolution information about the source structure and are sensitive to the large scale structure of the source while the longer baselines provide the high resolution information. GMRT array configuration was designed to have roughly half the antennas in a compact "Central Square" to provide the shorter spacings information, which is crucial mapping extended source and large scale structures in the sky. The uv-coverage of the central square antennas is shown in Figure 10.5. Notice that there are no measurements for ($u = 0, v = 0$). $V(0, 0)$ represents the total integrated flux received by the antennas and is absent in the visibility data. Effect of this on the image will be discussed later.

The astronomical coordinates depend on the line of intersection of the ecliptic and equatorial planes. The uv-coverage in turn depends on the position of the source in the astronomical coordinate system. Since the reference line of the this coordinate system changes because of the well known precession of the earth's rotation axis, the uv-coverage also becomes a function of the reference epoch for which the source position is specified. For the purpose of comparison and consistence in the literature, all source positions are specified in standard epochs (B1950 or J2000). Since each point in the (u, v, w) plane measures a particular spatial frequency and this spatial frequency coverage differs from one epoch to another, it's necessary to precess the source coordinates to the current epoch (also called the "date coordinates") prior to observations and all processing of the visibility data for the purpose of mapping must be done with (u, v, w) evaluated for the epoch of observations. Precessing the visibilities to the standard epoch prior to inverting the Eq. 10.2.10 will require specifying the real and imaginary parts of the visibility at

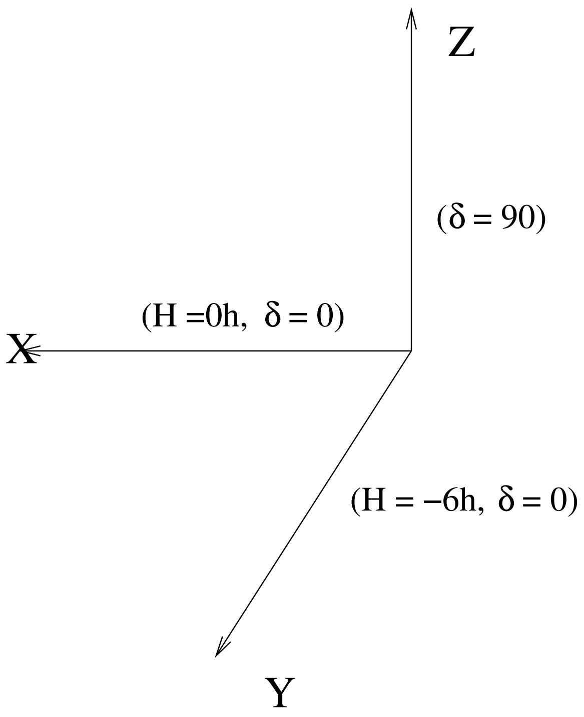

Figure 10.2: The (X,Y,Z) co-ordinate system used to specify antenna locations.

(u, v, w) coordinates which are in fact not measured (since the uv-coverage changes with epoch) introducing errors in the mapping procedure.

10.2 2D Relation Between Sky and Aperture Planes

Below, we derive the generalized 2D Fourier transform relation between the visibility and the source brightness distribution in the (u, v, w) system. The geometery for this derivation is shown in Fig 10.3.

Let the vector $\bar{L}_o$ represent the direction of the phase center and the vector $\bar{D}_\lambda$ represent the location of all antennas of an array with respect to a reference antenna. Then $\tau_g = \bar{D}_\lambda.\bar{L}_o$. Note that $2\pi\bar{D}_\lambda.\bar{L}_o = 2\pi w$ is phase by which the visibility should be rotated to stop the fringe. For any source in direction $\bar{L} = \bar{L}_o + \bar{\sigma}$, the output of an interferometer after fringe stopping will be

$$V(\bar{D}_\lambda) = \int_{4\pi} I(\bar{L})B(\bar{L})e^{2\pi\iota\bar{D}_\lambda.(\bar{L}-\bar{L}_o)}d\Omega. \tag{10.2.8}$$

The vector $\bar{L} = (l, m, n)$ is in the sky tangent plane, $\bar{L}_o$ is the unit vector along the w axis and $\bar{D}_\lambda = (u, v, w)$. The above equation can then be written as

$$V(u, v, w) = \int\int I(l, m)B(l, m)e^{2\pi\iota(ul+vm+w(\sqrt{1-l^2-m^2}-1))}\frac{dldm}{\sqrt{1-l^2-m^2}}. \tag{10.2.9}$$

If the array is such that all antennas are exactly located in the (u, v) plane, w is exactly zero and the above equation reduces to an exact 2D Fourier transform relation between

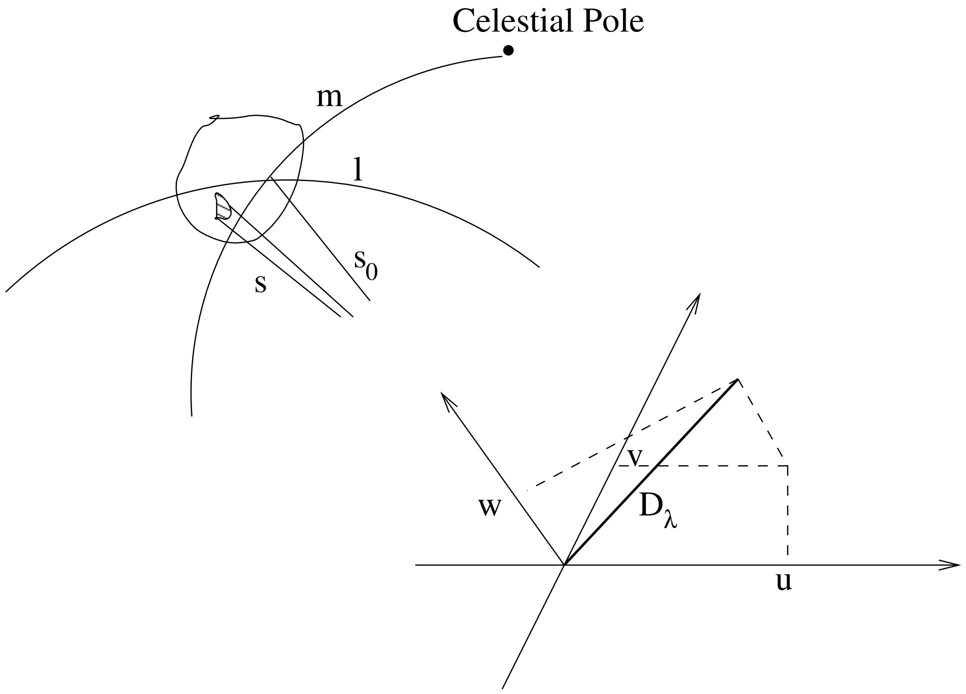

Figure 10.3: Relationship between the (l,m) co-ordiantes and the (u,v,w) co-ordinates

the source brightness distribution and the visibility. This is true for a perfect east-west array (like WSRT or ATCA). However to maximize the uv-coverage arrays like GMRT or VLA are not perfectly east-west. As mentioned earlier, the integrals in the above equation are finite for a small portion of the sky (being limited by the primary beam patter of the antennas). If the field of view being mapped is small (ie. for small l and m) $\sqrt{1-l^2+m^2}-1 \approx -\frac{1}{2}(l^2+m^2)$ and can be neglected. Eq. 14.1.1 becomes

$$V(u,v,w) \approx V(u,v,0) = \int\int I(l,m)B'(l,m)e^{2\pi\iota(ul+vm)}dldm. \qquad (10.2.10)$$

where $B' = B/\sqrt{1-l^2-m^2}$. Neglecting the w-term puts restrictions on the field of view that can be mapped without being effected by the phase error which is approximately equal to $\pi(l^2+m^2)w$. Eq. 10.2.10 shows the 2D Fourier transform relation between the surface brightness and visibility.

Since there are finite number of antennas in an aperture synthesis array, the uv-coverage is not continuous. Let

$$S(u,v) = \begin{matrix} 1, \text{ for all measured } (\mathbf{u},\mathbf{v}) \text{ points} \\ 0, \text{ every where else.} \end{matrix} \qquad (10.2.11)$$

Then to represent the real life situation, assuming that $B(l,m)=1$ over the extent of the source, Eq. 10.2.10 becomes

$$V(u,v)S(u,v) = \int\int I(l,m)e^{2\pi\iota(ul+vm)}dldm. \qquad (10.2.12)$$

Inverting the above equation and using the convolution theorem, we get $I^D = I * DB$ where DB is the Fourier transform of S. DB is the transfer function of the the telescope for imaging and is referred to as the *Dirty Beam*. I^D represents the raw image produced by an earth rotation aperture synthesis telescope and is referred to as the *Dirty Map*. Contribution of *Dirty Beam* to the map and methods of removing these these effects will be discussed in greater detail in later lectures.

In all the above discussion, we have assumed the observations are monochromatic with negligible frequency bandwidth and that the (u, v) measurements are instantaneous measurements. None of these assumptions are true in real life. Observations for continuum mapping are made with as large a frequency bandwidth as possible (to maximize the sensitivity) and the data is recorded after finite integration. Both result into degradation in the map plane and these effects will be discussed in the later chapters.

Neglecting the w-term essentially implies that the source brightness distribution is approximated to be restricted to the tangent plane at the phase center in the sky rather than on the surface of the celestial sphere. At low frequencies, where the antenna primary beams are larger and the radio emission from sources is also on a larger scale, this assumption restricts the mappable part of the sky to a fraction of the primary beam. Methods to relax this assumption will also be discussed in a later lecture.

10.3 Further Reading

1. Interferometry and Synthesis in Radio Astronomy; Thompson, A. Richard, Moran, James M., Swenson Jr., George W.; Wiley-Interscience Publication, 1986.

2. Synthesis Imaging In Radio Astronomy; Eds. Perley, Richard A.,Schwab, Frederic R., and Bridle, Alan H.; ASP Conference Series, Vol 6.

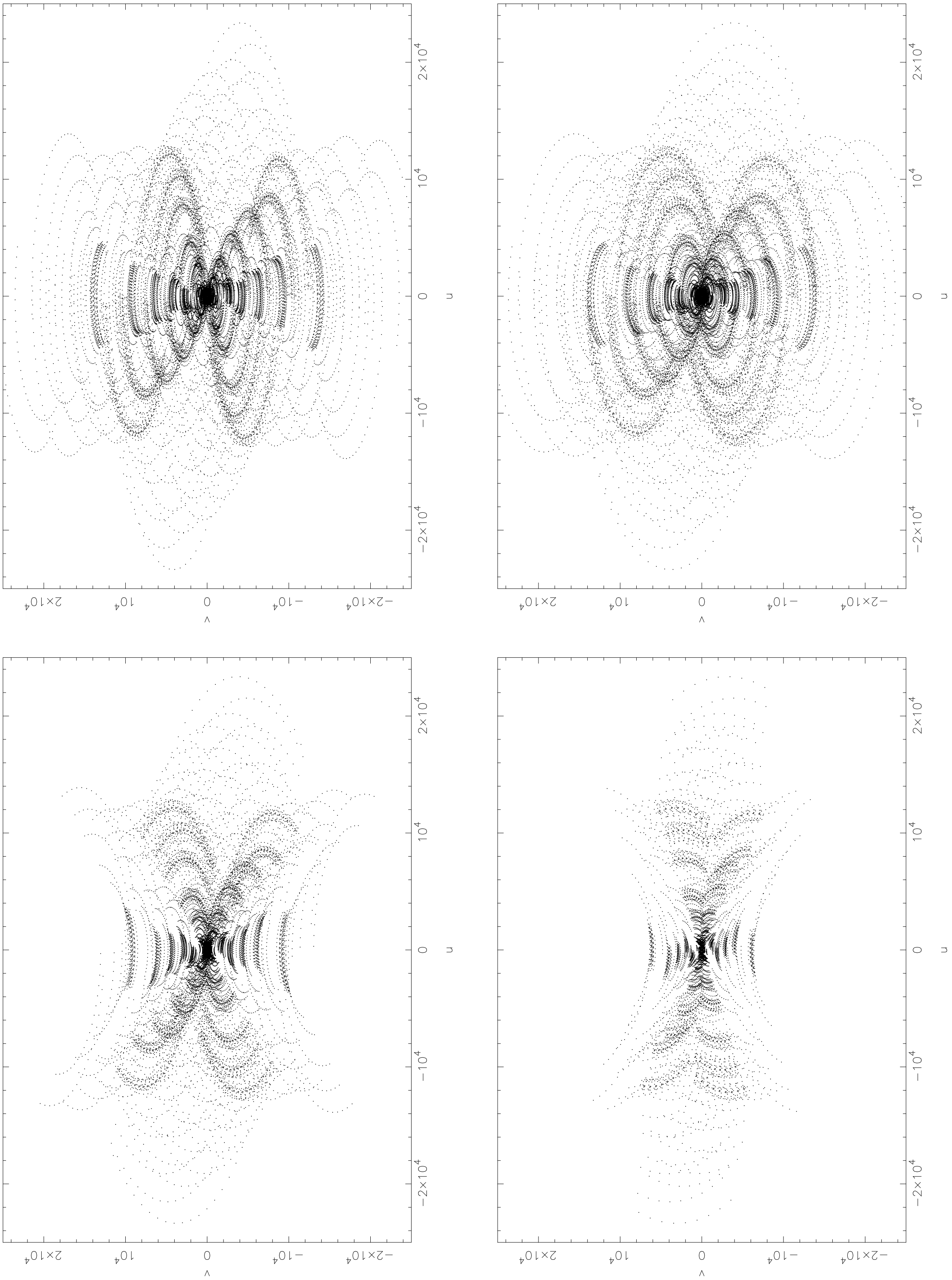

Figure 10.4: uv-coverage for a 10^h synthesis with the full GMRT array at δ of 19^o, 30^o, -30^o and -45^o. The u and v axes are in meters.

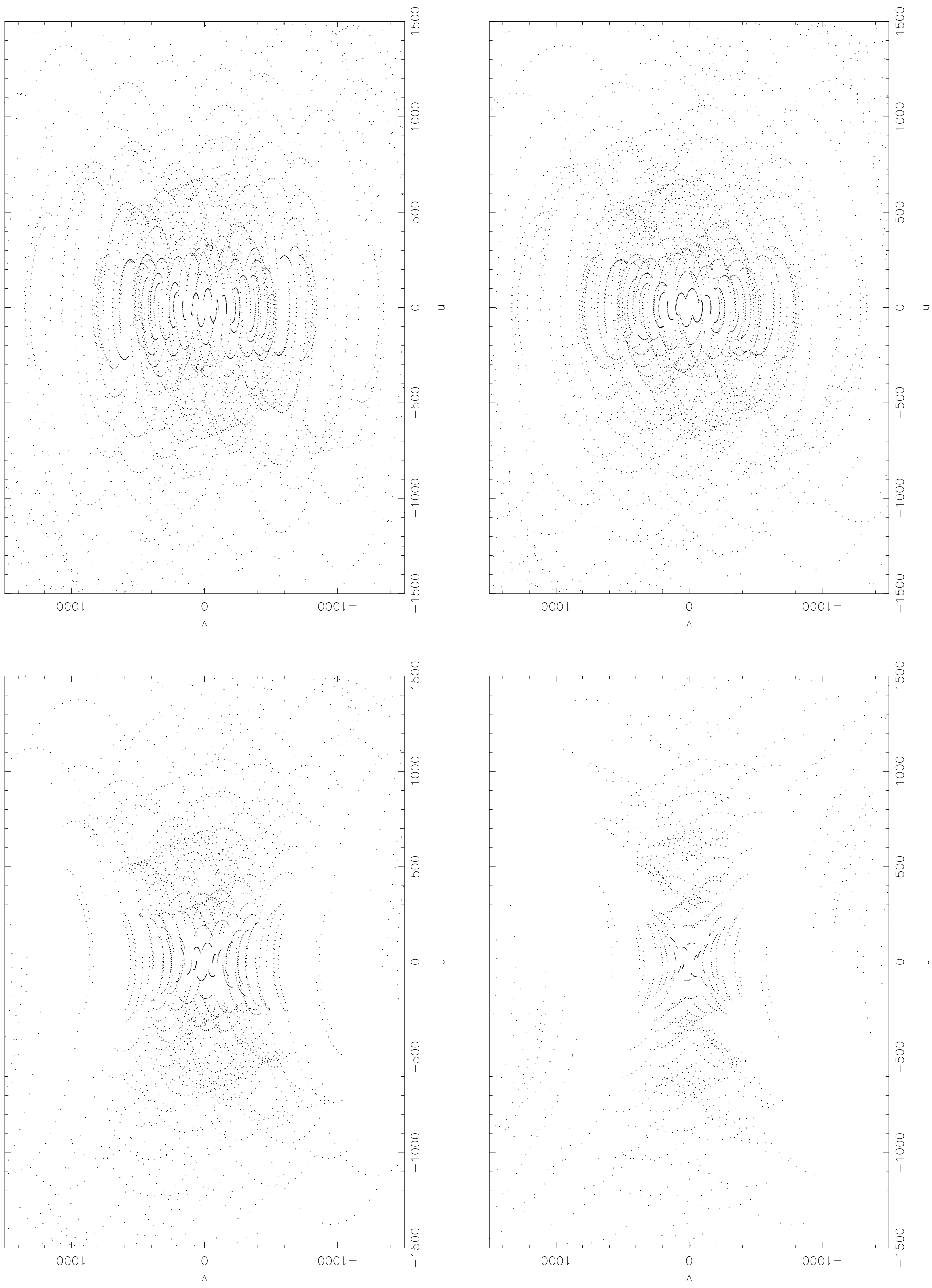

Figure 10.5: uv-coverage for a 10^h synthesis with GMRT Central Square at δ of 19^o, 30^o, -30^o and -45^o. The u and v axes are in meters.