POLARIMETRY

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– AUTHOR –

Introduction

Polarization is a important property of electromagnetic waves. In communications, completely polarized waves are used. In radio astronomy un-polarized components exist. In this lecture we shall describe the techniques to analyze polarization know as ***polarimetry***.

The complete polarization types of electromagnetic waves are:

(*i*) Linear Polarization.
(*ii*) Circular Polarization.
(*iii*) Elliptical Polarization.

Electromagnetic waves from of radio astronomical sources may posses:

(*i*) Random polarization (also known as un-polarized waves).
(*ii*) Partial polarization (completely polarized + un-polarized)

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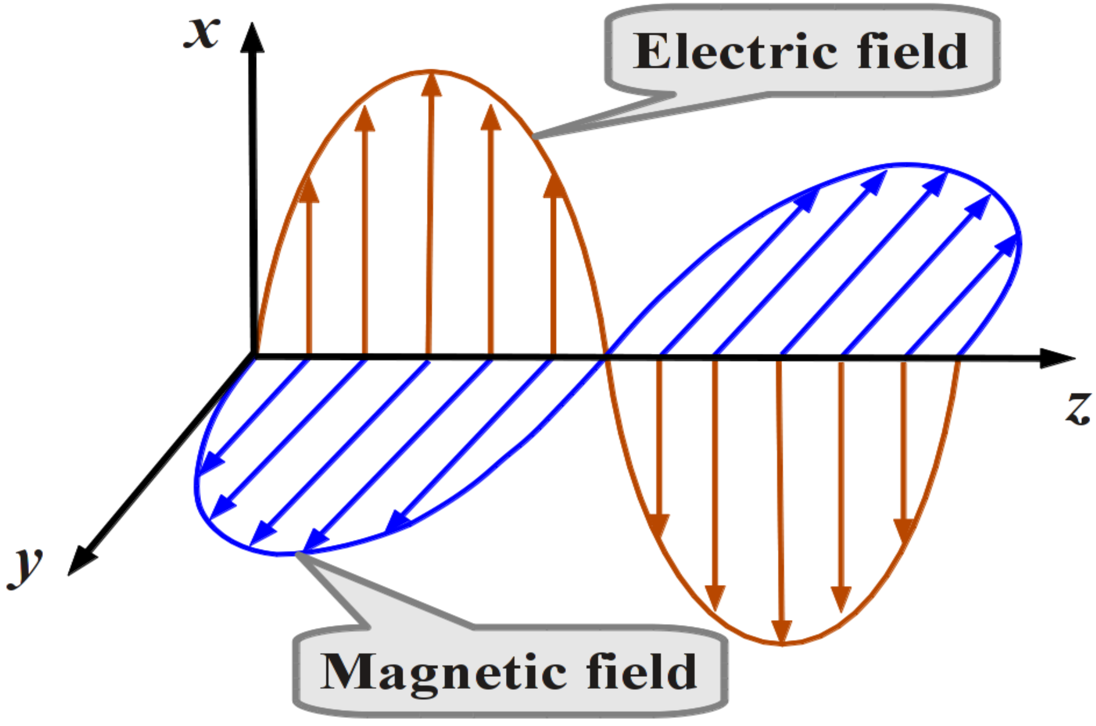
Light as transverse electromagnetic wave

The electric and magnetic fields of an electromagnetic wave are perpendicular to each other and transverse to the direction of propagation.



An electromagnetic wave is propagating along z-axis. Its electric field is aligned to the x-axis and magnetic field along the y-axis.

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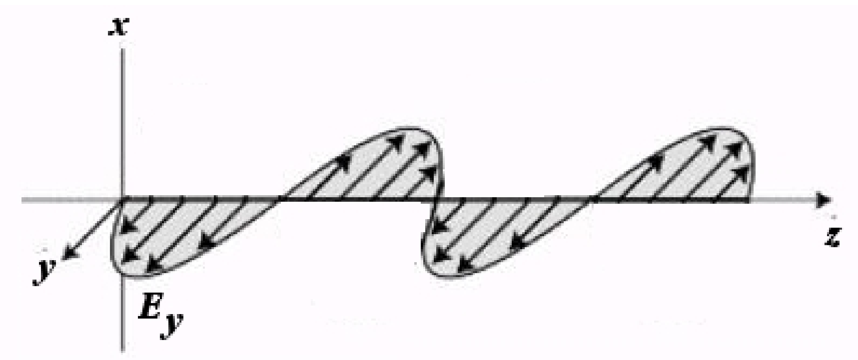
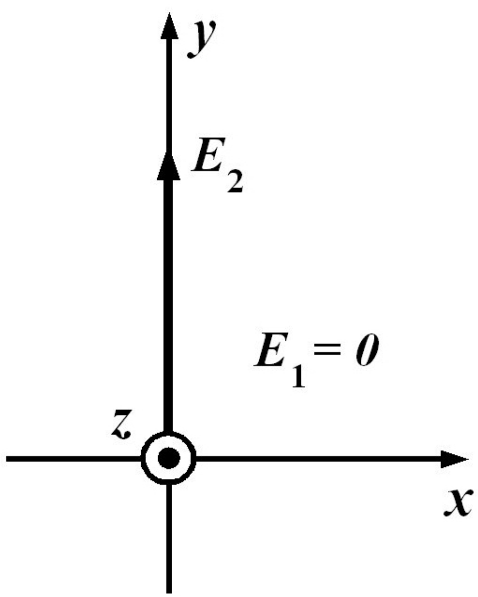
Linear Polarization

If the plane of the electric field does not change as the wave propagates, the wave is said to be ***linearly polarized****.*



Consider a wave to be propagating along *z*-axis, and the electric field lies only in the *y*-*z* plane. The wave is said to linearly polarized along *y*-axis.



The electric field components along *x* and *y* are:

$$E_x = 0$$

$$E_y = E_2 \cos(\omega t - \beta z)$$

where, angular frequency $\omega = 2\pi v$

and wave number $\beta = 2\pi/\lambda$

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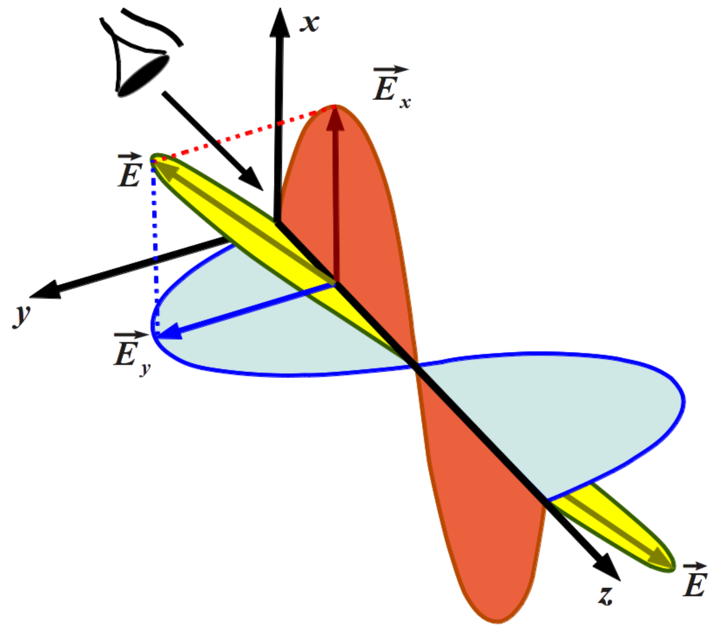
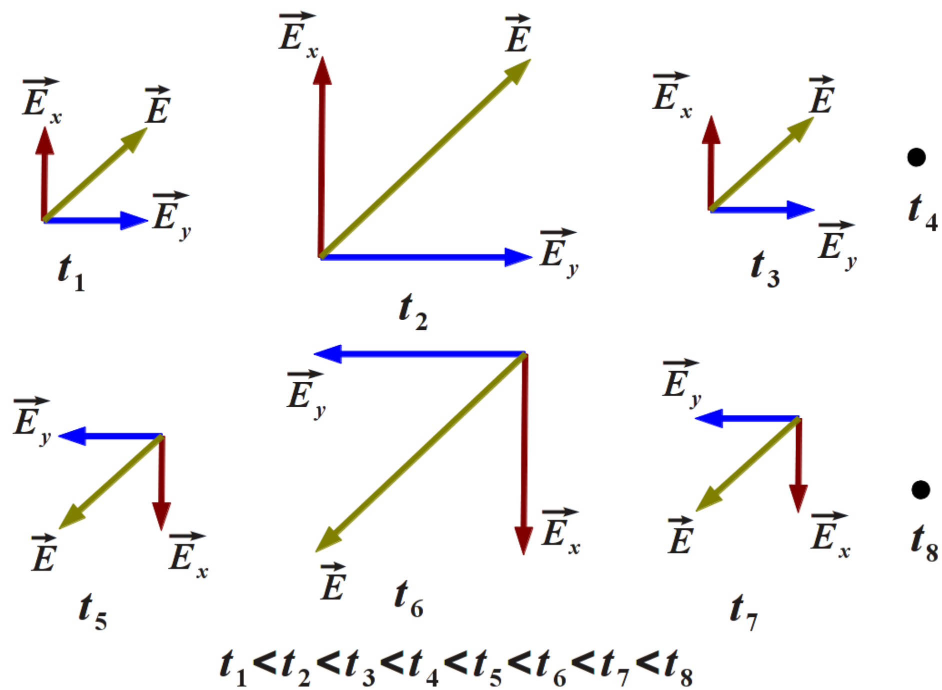
Components of a Linearly Polarized Wave

A propagating linearly polarized wave may be thought of as a vector sum of two waves in orthogonal planes propagating in same direction.



A propagating linearly polarized wave.



Electric field components as observed.

Electric field vector **E** is the vector sum of $\mathbf{E}_x$ and $\mathbf{E}_y$ components:

$\vec{E} = \vec{E_x} + \vec{E_y}$ where, $E_x = E_1 \sin(\omega t - \beta z)$ and, $E_y = E_2 \sin(\omega t - \beta z)$

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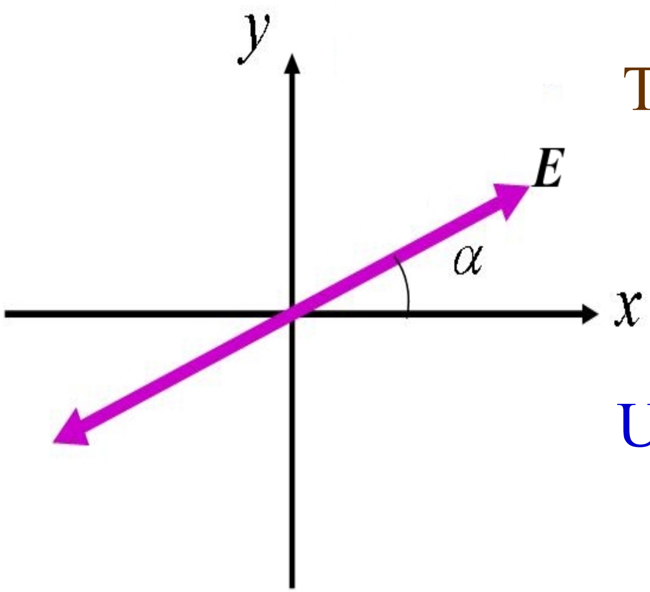
Components of a Linearly Polarized Wave

Consider a generic case where the electric field lies in a plane somewhere between *y*-*z* and *x*-*z* plane.



α = inclination of E w.r.t. x-axis.

The components of **E** along *x* and *y* are:

$$E_x = E\cos(\alpha)$$

$$E_y = E\sin(\alpha)$$

Using unit vectors along *x* and *y* we may write:

$$\vec{E} = \vec{e_x}\,E_x + \vec{e_y}\,E_y$$

The ratio of E_y to E_x is $\dfrac{E_y}{E_x} = \tan\alpha$

Note: For linear polarization,

(*i*) Phase difference between E_y and E_x must be zero.

(*ii*) Magnitudes of E_y and E_x can be different.

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Linear Polarization: Field Component Ratio

Examples:

(i) 0° linear polarization along x-axis:

$$E_y / E_x = 0,$$

(ii) 90° linear polarization along y-axis:

$$E_y / E_x = \infty$$

(iii) 45° linear polarization:

$$E_y / E_x = 1$$

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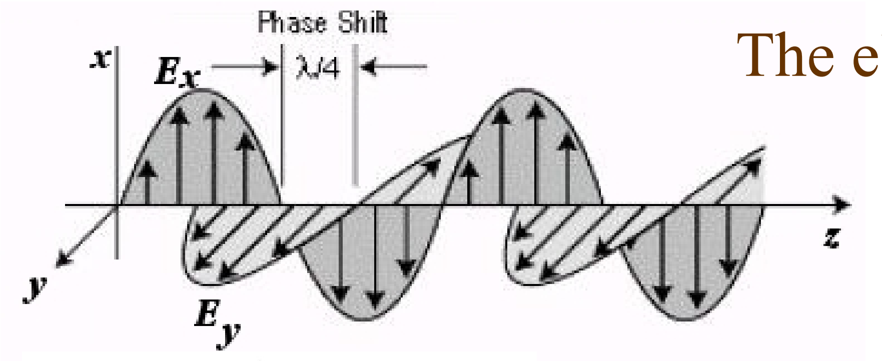
Circular Polarization

If the magnitudes of E_x and E_y are equal, but there exists a phase difference of π/2 *or* –π/2, *the tip of the electric field vector describes a circle and wave is said to be* ***circularly polarized***.



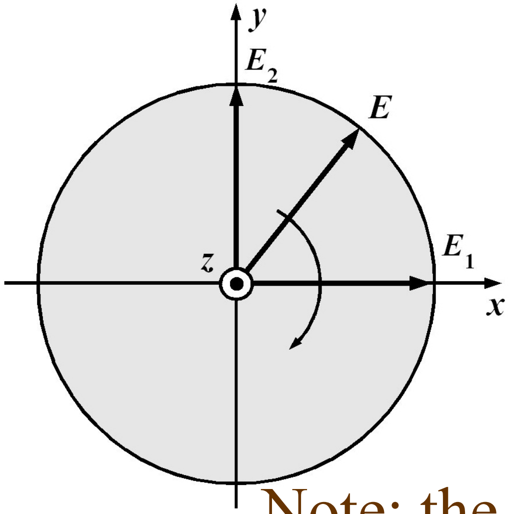
The electric fields can be expressed as:

$$E_x(z,t) = E_1 \sin(\omega t - \beta z)$$

$$E_y(z,t) = E_2 \sin(\omega t - \beta z + \delta)$$

where, $\delta = \pm \pi/2$

Hence, $E_y(z,t) = E_2 \cos(\omega t - \beta z)$



Consider the circle to be a special case of an ellipse.

Hence, axial ratio = $\dfrac{\text{Major axis}}{\text{Minor axis}} = \dfrac{|E_y|}{|E_x|} = \dfrac{E_2}{E_1} = 1$

Note: the axial ratio is always unity for circular polarization.

$$\vec{E} = \vec{e_x} E_1 \sin(\omega t - \beta z) + \vec{e_y} E_2 \sin(\omega t - \beta z + \pm\pi/2)$$ where, $E_1 = E_2$

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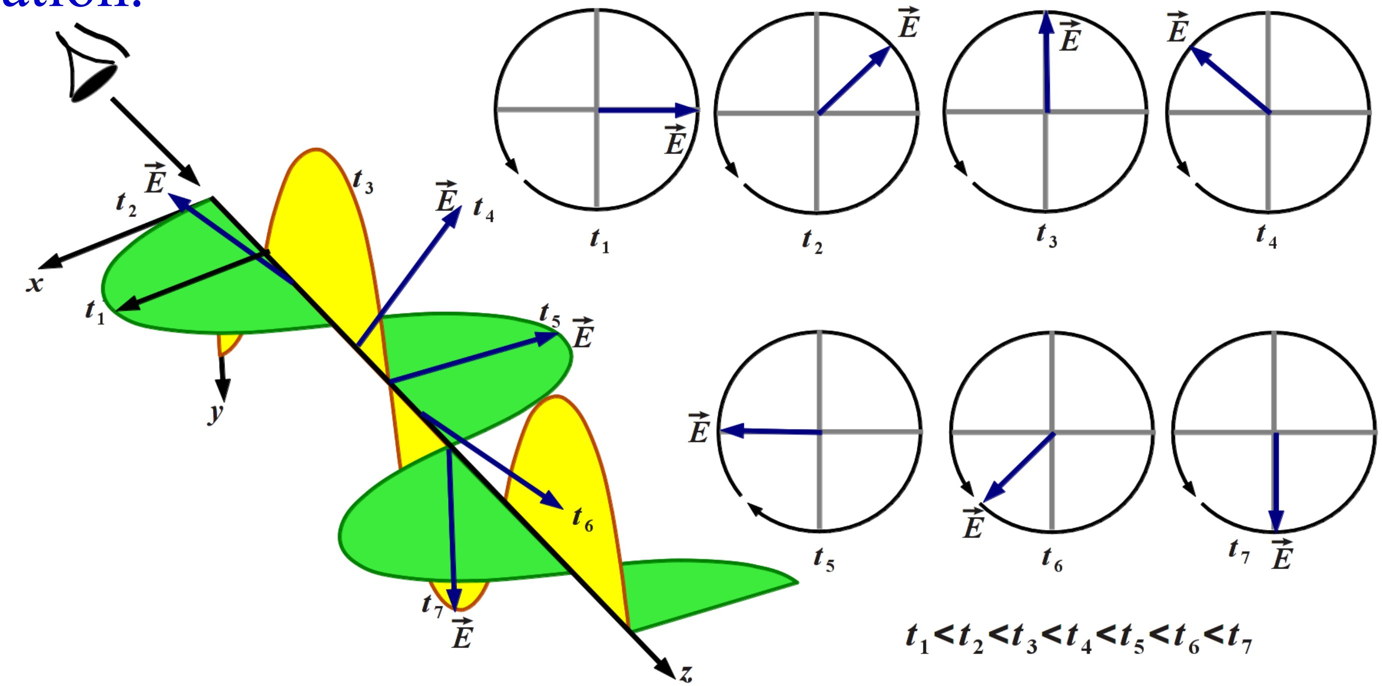
Right and Left Circular Polarizations

The EM waves can be either right circularly polarized or left circularly polarized depending upon rotation of the electric field along the direction of propagation.

If the electric field **E** rotates like a right handed screw as seen from the source, it is right circular.



$$E_x(z,t) = E_0 \sin(\omega t - \beta z) \text{ and } E_y(z,t) = E_0 \cos(\omega t - \beta z)$$

If the electric field E field rotates like a left handed screw as seen from the source, it is left circular.

$$E_x(z,t) = E_0 \sin(\omega t - \beta z) \text{ and } E_y(z,t) = -E_0 \cos(\omega t - \beta z)$$

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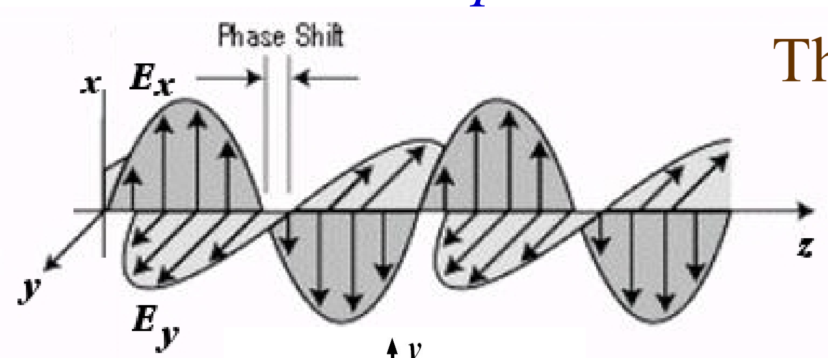
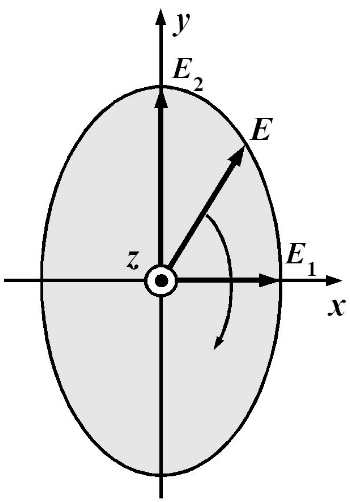
Elliptical Polarization

If the magnitudes of $\mathbf{E}_x$ *and* $\mathbf{E}_y$ *are not equal, and there exists a phase difference between the two, the tip of the electric field vector describes an ellipse and wave is said to be* ***elliptcally polarized***.





The electric fields can be expressed as:

$$E_x = E_1 \sin(\omega t - \beta z)$$

$$E_y = E_2 \sin(\omega t - \beta z + \delta)$$

where, $\delta \neq 0$

Also note that $|E_y| \neq |E_x|$ i.e., $\mathrm{E}_2 \neq \mathrm{E}_1$

$$\text{Axial ratio} = \frac{\text{Major axis}}{\text{Minor axis}} = \frac{|E_y|}{|E_x|} = \frac{\mathrm{E}_2}{\mathrm{E}_1} \neq 1$$

Generic electric field:

$$\vec{E} = \mathrm{e_x}\, E_1 \sin(\omega t - \beta z) + \mathrm{e_y}\, E_2 \sin(\omega t - \beta z + \delta)$$

Note: (*i*) Circular polarization is a special case of elliptical polarization where, $E_2 = E_1$, $\delta = \pm \pi/2$, and the axial ratio = 1.

(*ii*) Linear + Circular pol. = Elliptical polarization.

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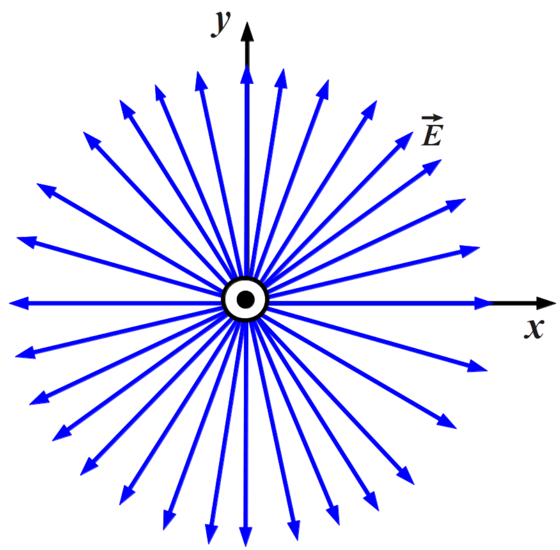
Random Polarization

If the plane of the electric field changes its orientation randomly but the magnitude is constant, it is known as ***randomly polarized*** *or* ***un-polarized*** *EM wave.*



Properties:

- The plane of the electric field (and so of the magnetic field) are random functions of time.
- The probability of orientation of the electric field in any direction in the *x*-*y* plane is same.
- The magnitude of the electric field (and so of the magnetic field) at any instant of time is always same.

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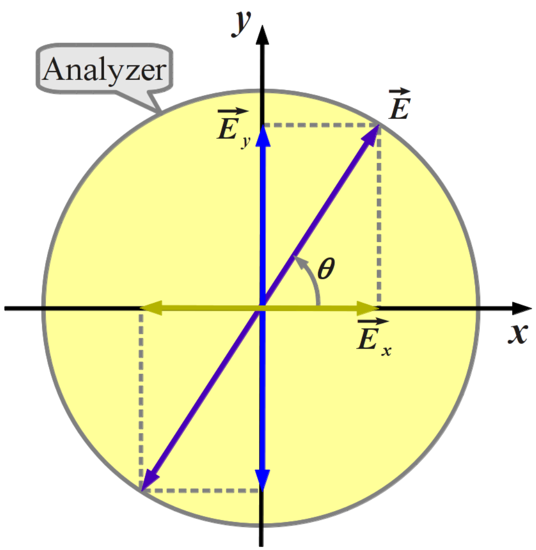
Polarization Extraction and Malus' Law

It is possible to extract one or more linear components of polarization from a randomly polarized wave using polarizers.



A polarized wave **E** at angle θ (obtained from a unpolarized source using a polarizer) enters an analyzer.

Case (*i*): If the analyzer is horizontal, component of **E** along x is $E_x = E_0 \cos \theta$.

Case (*ii*): If the analyzer is vertical, component of **E** along y is $E_y = E_0 \sin \theta$.

Intensity I is related to **E** as $I = E^* E$, where E^* is complex conjugate of E.

Malus' law: *The intensity I of a polarized light when passed through a linear polarizer (analyzer) making an angle θ with the incoming light's polarization plane is* $I = I \cos^2 \theta$.

Using Malus' law, we may relate the x and y components of intensities with electric field of incoming wave as:

$$I_y = E_0^2 \sin^2 \theta \text{ and } I_x = E_0^2 \cos^2 \theta.$$

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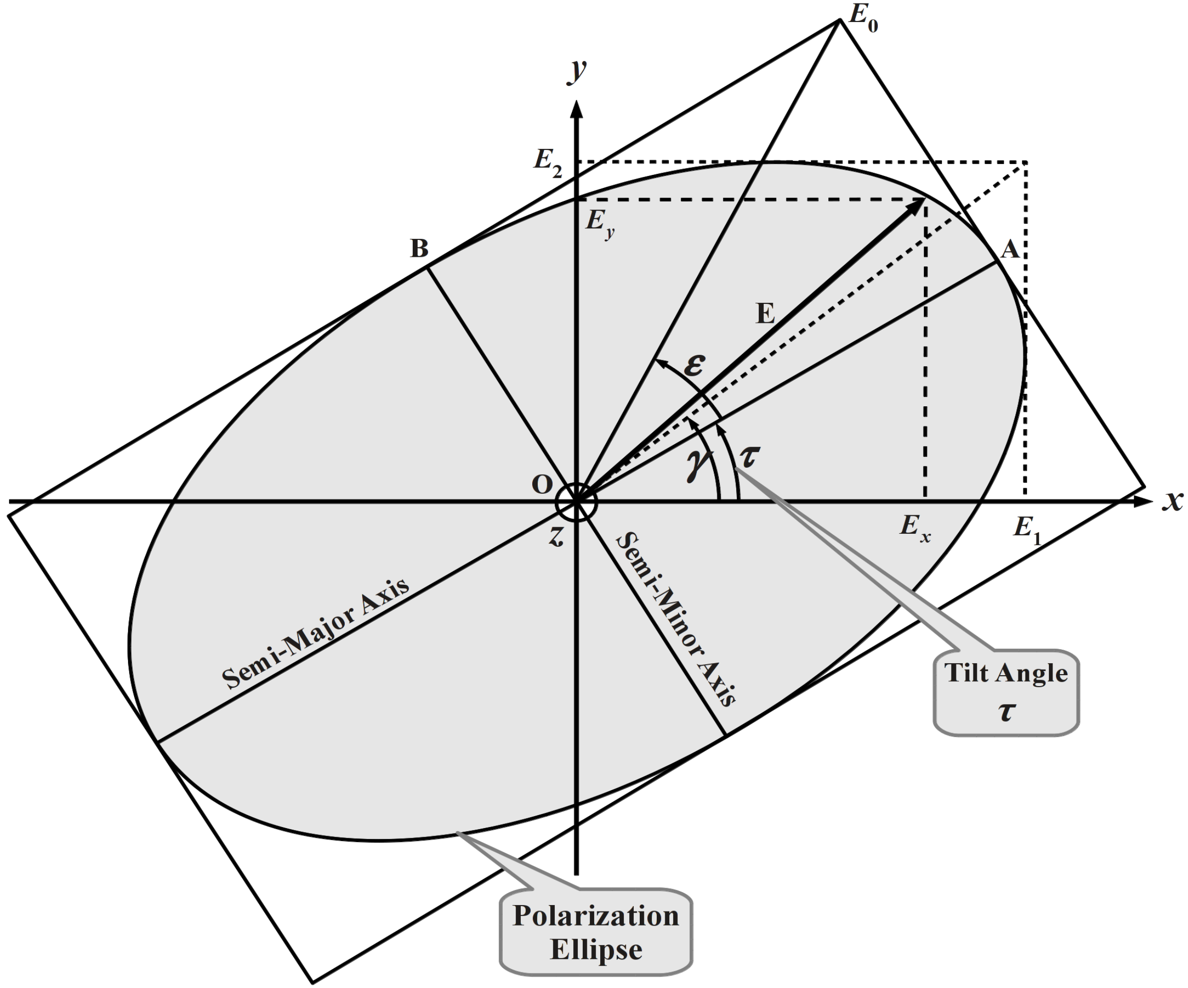
Polarization Ellipse

Any form of complete polarization resulting from a coherent source can be analyzed using polarization ellipse. Let the ellipse (described by the tip of electric field) be tilted at an angle τ with respect to x-axis. We construct a diagram to relate the electric field components with x and y coordinates as shown. The values of E_1 and E_2 are fixed from ellipse geometry which will be used in next derivations. We may choose the generic form of **E** as: $\vec{E} = \hat{x}\, E_1 \sin(\omega t - \beta z) + \hat{y}\, E_2 \sin(\omega t - \beta z + \delta)$



where, δ is the phase difference between E_x and E_y.

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Polarization Ellipse

We have choosen the generic form of **E** as:

$$\overrightarrow{E} = \widehat{\mathrm{x}}\, E_1 \sin(\omega t - \beta z) + \widehat{\mathrm{y}}\, E_2 \sin(\omega t - \beta z + \delta)$$

Then in the *x*-*y* plane at z = 0, we have:

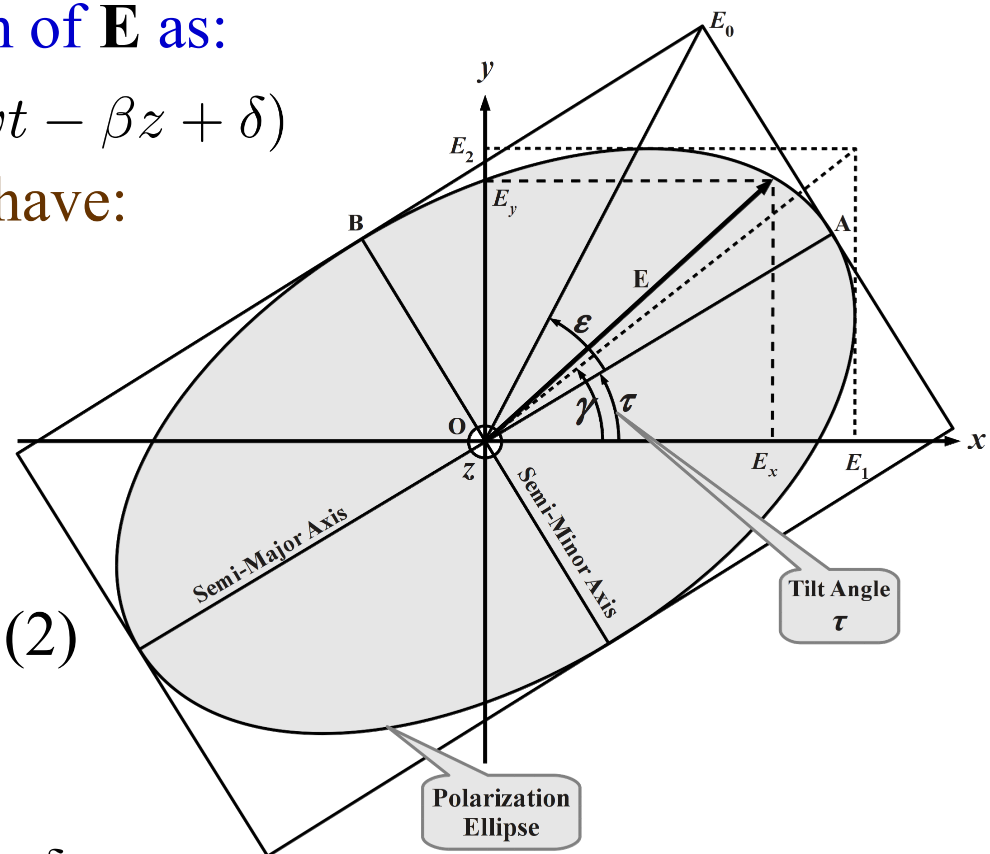
$$E_x = E_1 \sin \omega t$$

or $\quad \sin \omega t = \dfrac{E_x}{E_1} \qquad \ldots(1)$

and $\cos \omega t = \sqrt{1 - \left(\dfrac{E_x}{E_1}\right)^2} \quad \ldots(2)$

Also at z = 0, $E_y = E_2 \sin(\omega t + \delta)$

or $\quad \dfrac{E_y}{E_2} = \sin \omega t\ \cos \delta + \cos \omega t\ sin\delta \qquad \ldots(3)$



Substituting equations (1) and (2) in (3) we get:

$$\frac{E_y}{E_2} = \frac{E_x}{E_1} \cos \delta + \sqrt{1 - \left(\frac{E_x}{E_1}\right)^2}\ \sin \delta \qquad \ldots(4)$$

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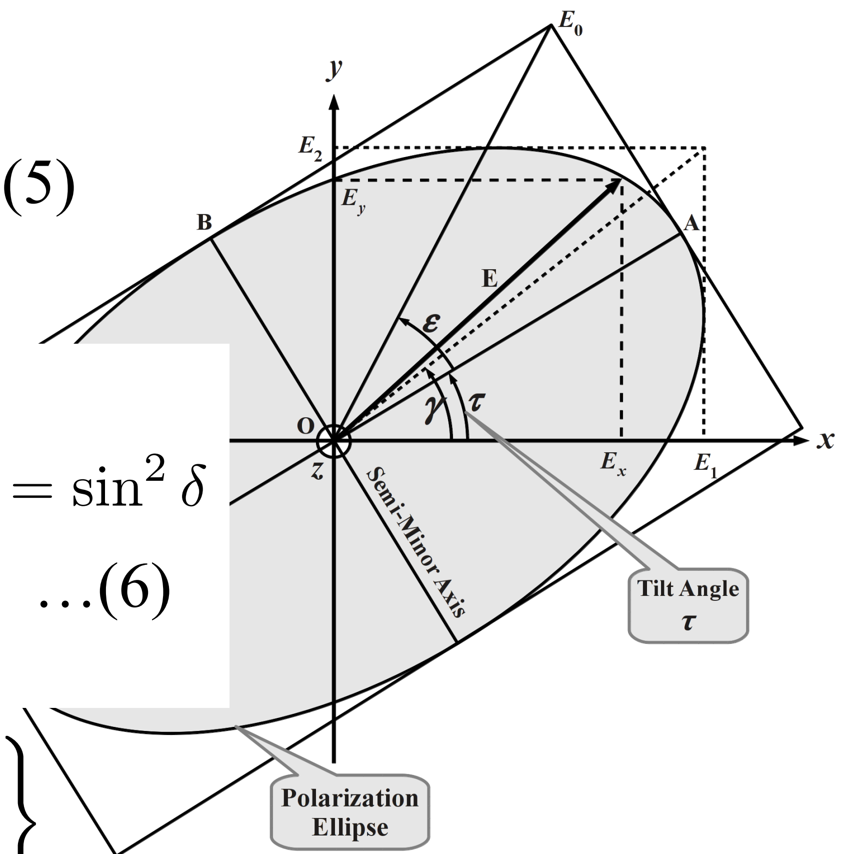
Polarization Ellipse

Rearranging equation (4) we get:

$$\sqrt{1-\left(\frac{E_x}{E_1}\right)^2}\,\sin\delta = \frac{E_y}{E_2} - \frac{E_x}{E_1}\cos\delta \quad \ldots(5)$$

Squaring and simplifying we get:

$$\left(\frac{E_x}{E_1}\right)^2 - 2\left(\frac{E_x}{E_1}\right)\left(\frac{E_y}{E_2}\right)\cos\delta + \left(\frac{E_y}{E_2}\right)^2 = \sin^2\delta \quad \ldots(6)$$



We may re-write the above equation as:

$$\left.\begin{array}{l} aE_x^2 + bE_xE_y + cE_y^2 = 1 \text{ where we have :} \\ a = \dfrac{1}{E_1^2\sin^2\delta} \quad b = \dfrac{2\cos\delta}{E_1E_2\sin^2\delta} \quad c = \dfrac{1}{E_2^2\sin^2\delta} \end{array}\right\} \ldots(7)$$

Equation (7) describes the polarization ellipse. The axial ratio AR is ratio of length of major axis to length of minor axis as given below:

$$AR = \frac{OA}{OB}, \quad 1 \le AR \le \infty \quad \ldots(8)$$

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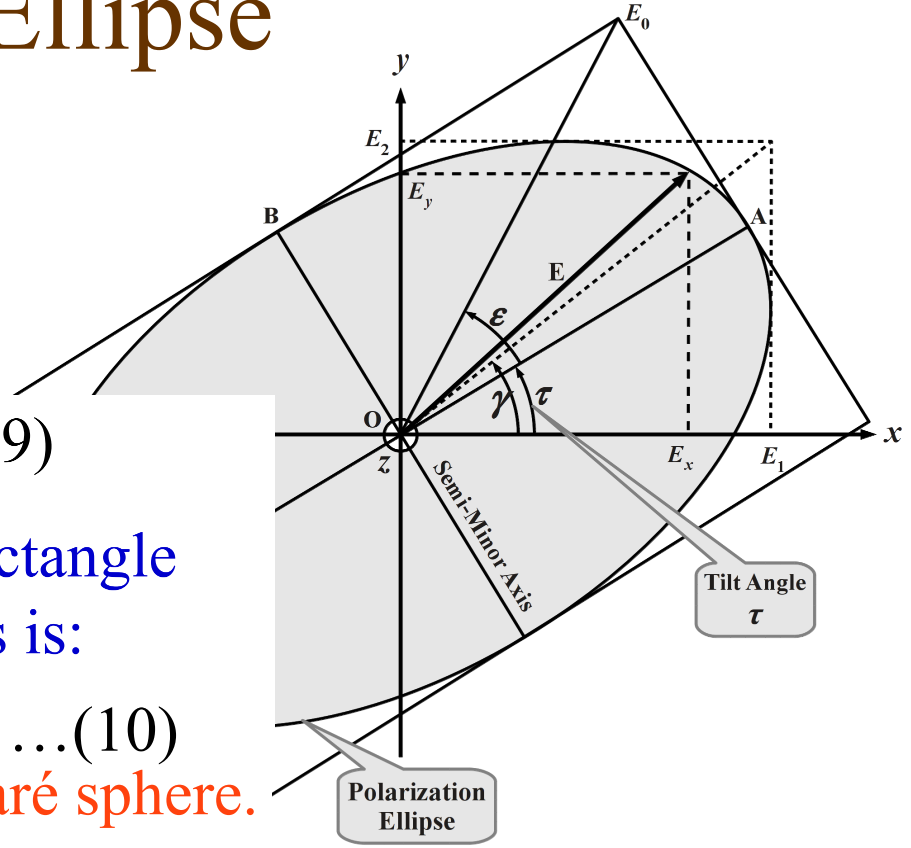
Polarization Ellipse

$$AR = \frac{OA}{OB}, \quad 1 \le AR \le \infty \quad \ldots(8)$$

From the geometry, we get the expressions for other angles as:

$$\gamma = \tan^{-1}\left(\frac{E_2}{E_1}\right), \quad 0^\circ \le \gamma \le 90^\circ \quad \ldots(9)$$



Angle ε between top right corner of rectangle enclosing the ellipse and its major axis is:

$$\varepsilon = \cot^{-1}(\pm AR), \quad -45^\circ \le \varepsilon \le +45^\circ \quad \ldots(10)$$

Note: ε, γ and τ will be used in Poincaré sphere.

$$\left.\begin{aligned} S_{\text{av}} &= \hat{z}\frac{1}{2}\left(\frac{E_1^2+E_2^2}{Z_0}\right) = \hat{z}\frac{1}{2}\left(\frac{E^2}{Z_0}\right) \\ \text{where, } E &= \sqrt{E_1^2+E_2^2} \end{aligned}\right\} \ldots(11)$$

Here, Z_0 is intrinsic impedance of free space.

Typical values of E_x and E_y for linear and circular polarization using the polarization ellipse are tabulated in next page..

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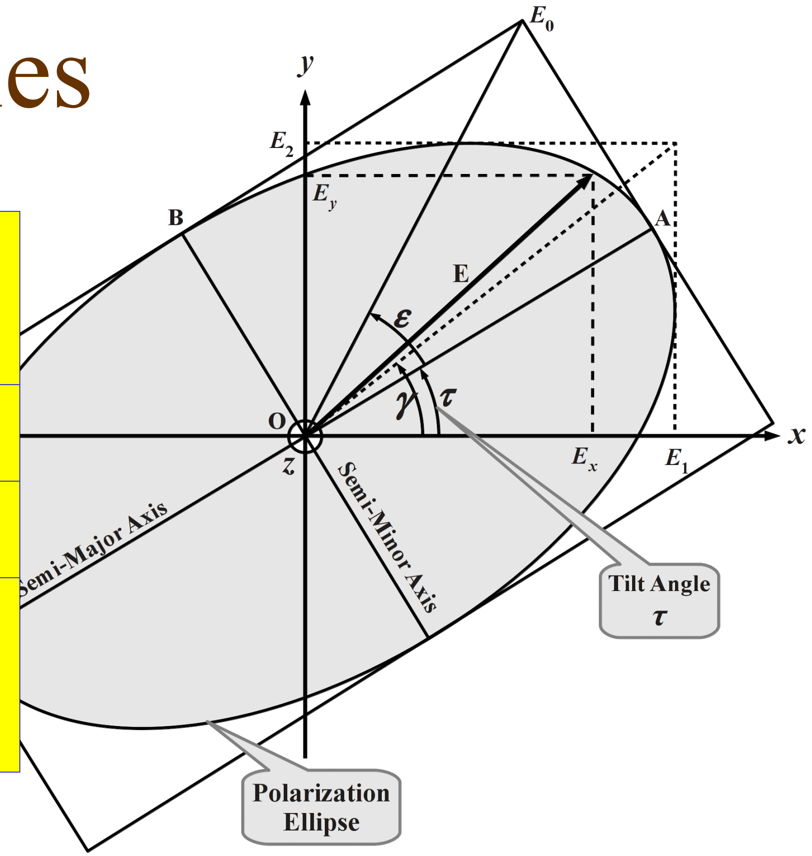
Polarization Examples

Linear Polarization

Polarization Axis	E_x	E_y	τ	δ
y-axis	0	> 0	90º	-
x-axis	> 0	0	0º	-
45° with x and y axes	$E_2 = E_1$	$E_2 = E_1$	45º	0º



Circular Polarization

Type of polarization	E_x	E_y	τ	δ
Left Circular (IEEE standards)	$E_2 = E_1$	$E_2 = E_1$	-	+90º
Right Circular (IEEE standards)	$E_2 = E_1$	$E_2 = E_1$	-	-90º

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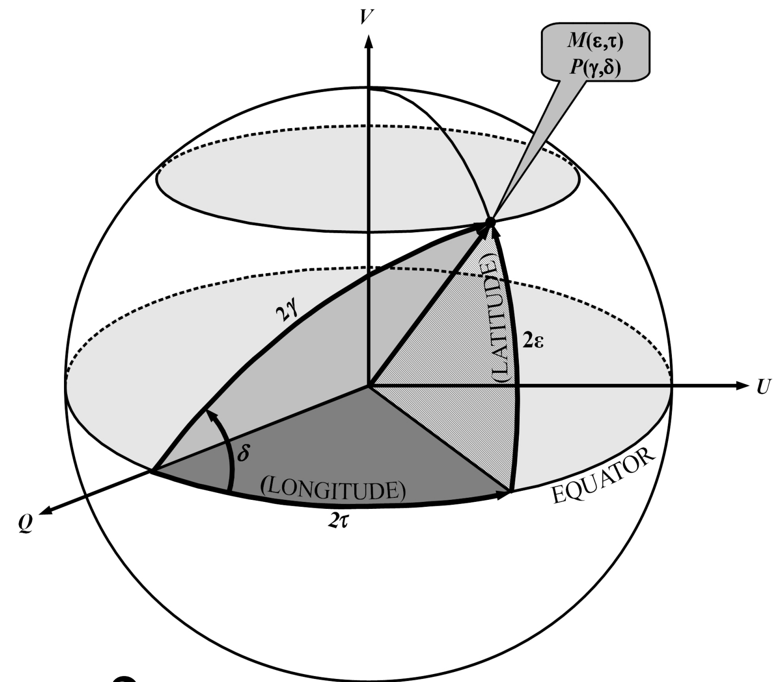
Poincaré Sphere and Polarization Ellipse

The Poincaré sphere is a another graphical representation of the state of polarization. It uses a spherical coordinate system. The state of polarization is described by a point on the sphere, where the longitude and latitude of the point are related to the polarization ellipse.



$$\text{Longitude} = 2\tau \qquad \text{Latitude} = 2\varepsilon$$

$$\text{Great circle angle} = 2\gamma \qquad \text{Equator to great circle angle} = \delta$$

For deriving the Stokes parameters *I*, *Q*, *U* and *V* later in this lecture, the cartesian coordinate system is also shown with axis marked *Q*, *U* and *V*.

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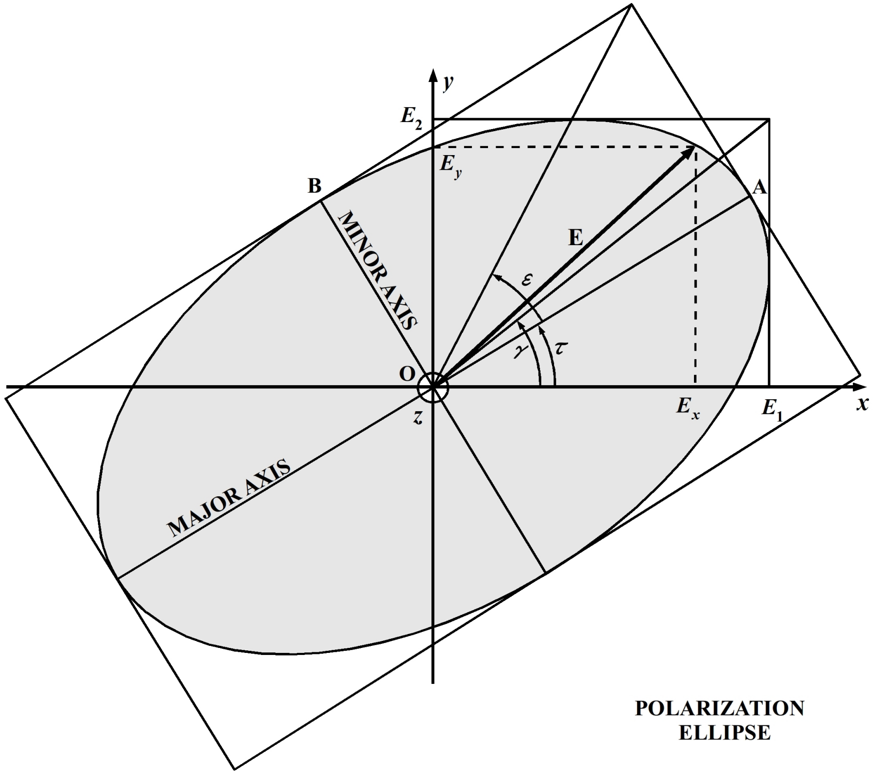
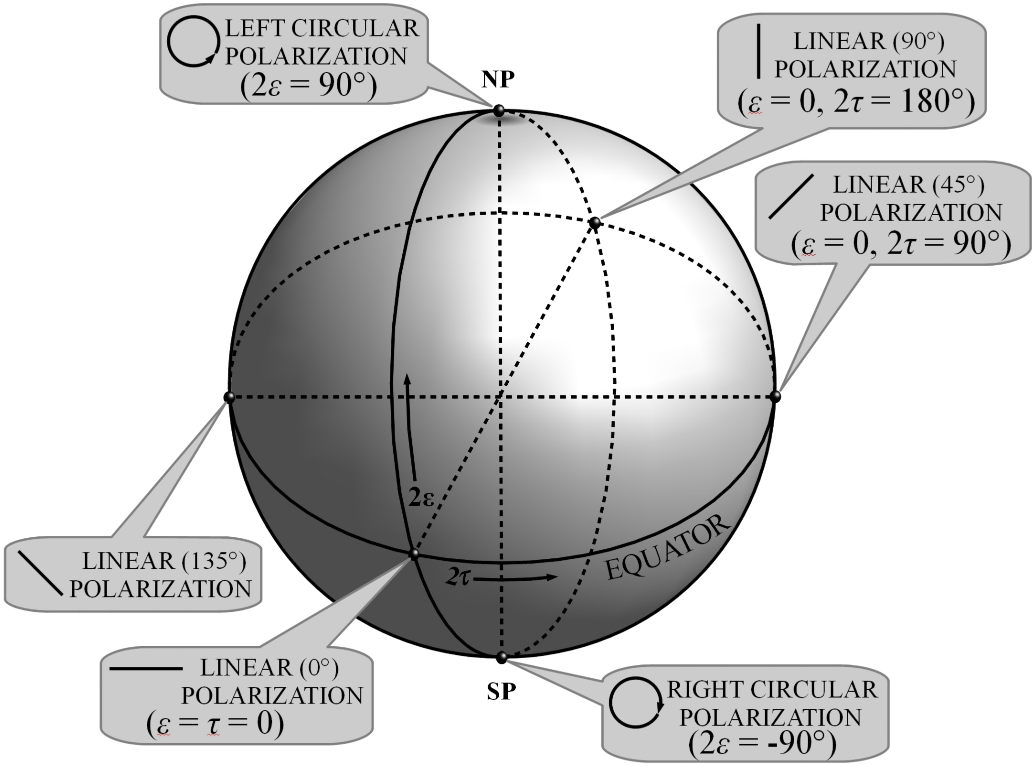
Poincaré Sphere and Polarization Ellipse





$$\cos 2\gamma = \cos 2\varepsilon \; \cos 2\tau$$

$$\tan \delta = \frac{\tan 2\varepsilon}{\sin 2\tau}$$

$$\tan 2\tau = \tan 2\gamma \; \cos \delta$$

$$\sin 2\varepsilon = \sin 2\gamma \; \sin \delta$$

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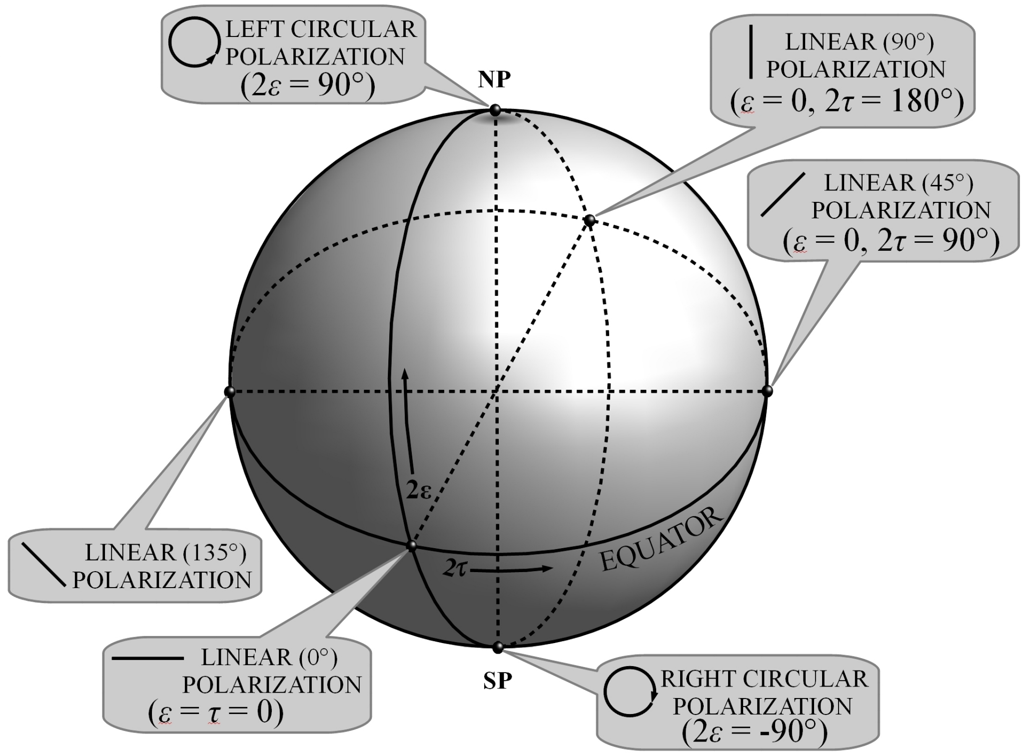
Poincaré Sphere and Polarization Ellipse

Case-1:

For $\delta = 0$, or for $\delta = \pm 180^o$, E_x and E_y are either in exact phase or out of phase. Hence, any point on the equator represents a state of linear polarization.



At the origin (ε=0, τ=0) which is same as (δ=0, γ=0), the polarization is linear and horizontal. At (ε=0, 2τ=90^o), the polarization is linear with a tilt angle of 45^o linear and vertical.

At (ε=0, 2τ=180^o), the polarization is linear and vertical. Similarly, the polarization is 135^o at (ε=0, 2τ=270^o) and so on.

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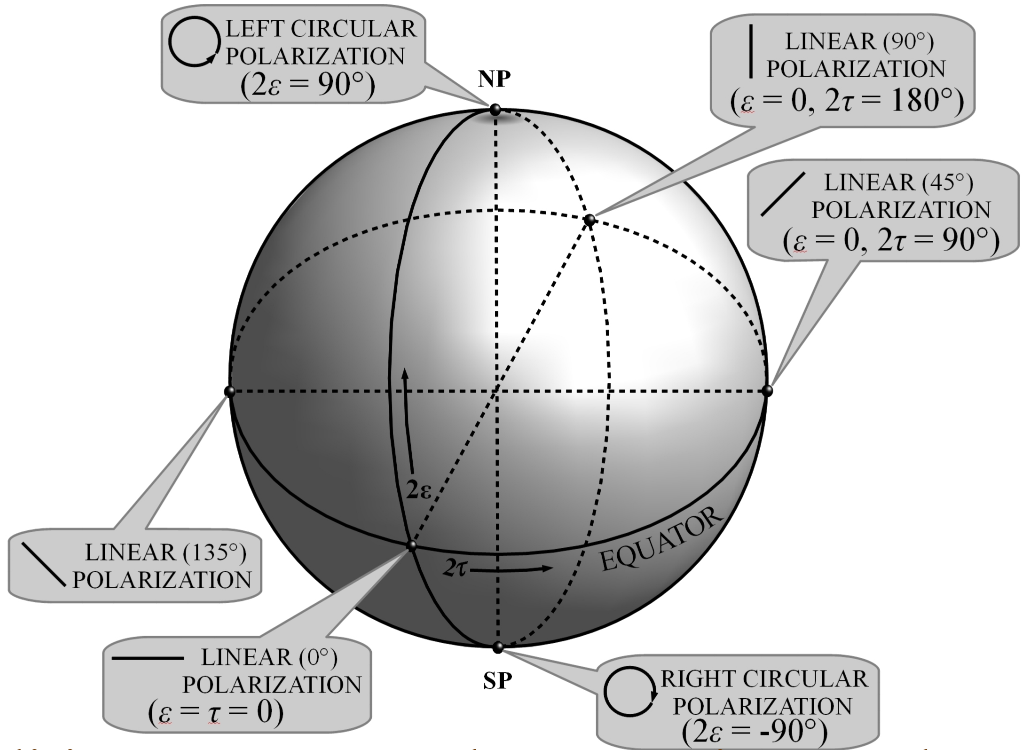
Poincaré Sphere and Polarization Ellipse

Case-2:

At $\delta = \pm 90^o$, i.e., at the poles ($2\varepsilon = \pm 90^o$), the polarizations are circular. The North pole represents the left circular polarization and the South pole represents the right circular polarization.



Case-3:

Cases 1 and 2 are limiting conditions. In general, any point on the northern hemisphere describes a left elliptically polarized wave ranging from a pure left circular polarization at the pole to linear polarization at the equator. Similarly, any point on the southern hemisphere describes a right elliptically polarized wave ranging from a pure right circular polarization at the pole to linear polarization at the equator.

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Partial Polarization & Stokes Parameters

So far we have analyzed only monochromatic waves (single frequency) where, E_1, E_2 and δ were assumed to be constants or slowly varying. We have also introduced random polarization and Malus' law. We shall now use these concepts and tools for analyzing the celestial radio sources.

The celestial radio sources emit over a wide frequency range. Within a finite bandwidth $\Delta\nu$, the signal consists of a superposition of numerous statistically independent waves possessing variety of polarizations (randomly polarized). In most general situations, the wave is partially polarized which may be regarded as the sum of two components: (*i*) a completely polarized component, and (*ii*) a randomly polarized component. Such waves can be analyzed using ***Stokes parameters*** which we introduce next.

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Stokes Parameters

Limitations of Polarization ellipse

Since E_x and E_y are instantaneous time functions, the polarization ellipse is valid only for a given instant of time. For an un-polarized wave, it is not possible to obtain the instantaneous tilt angle τ or the axial ratio.

To overcome these limitations ***Stokes parameters*** are used, which is a set of four time averaged components (I, Q, U, V) obtained from the wave and can be related to the polarization ellipse. From its properties, (*i*) if two waves have identical Stokes parameters, the waves are identical, and (*ii*) if several independent waves propagating in the same direction are superimposed, the Stokes parameters of the resultant wave is the sum of Stokes parameters of individual waves.

We shall first investigate them for the following three cases:

(*i*) Completely polarized waves.

(*ii*) Completely un-polarized waves.

(*iii*) Partially polarized waves.

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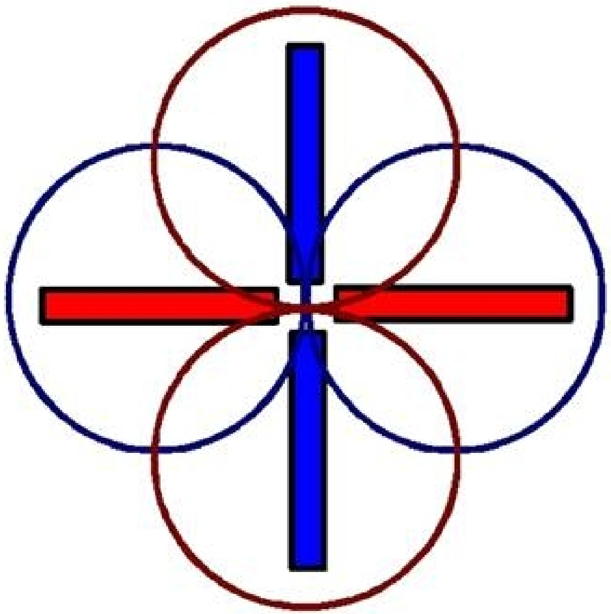
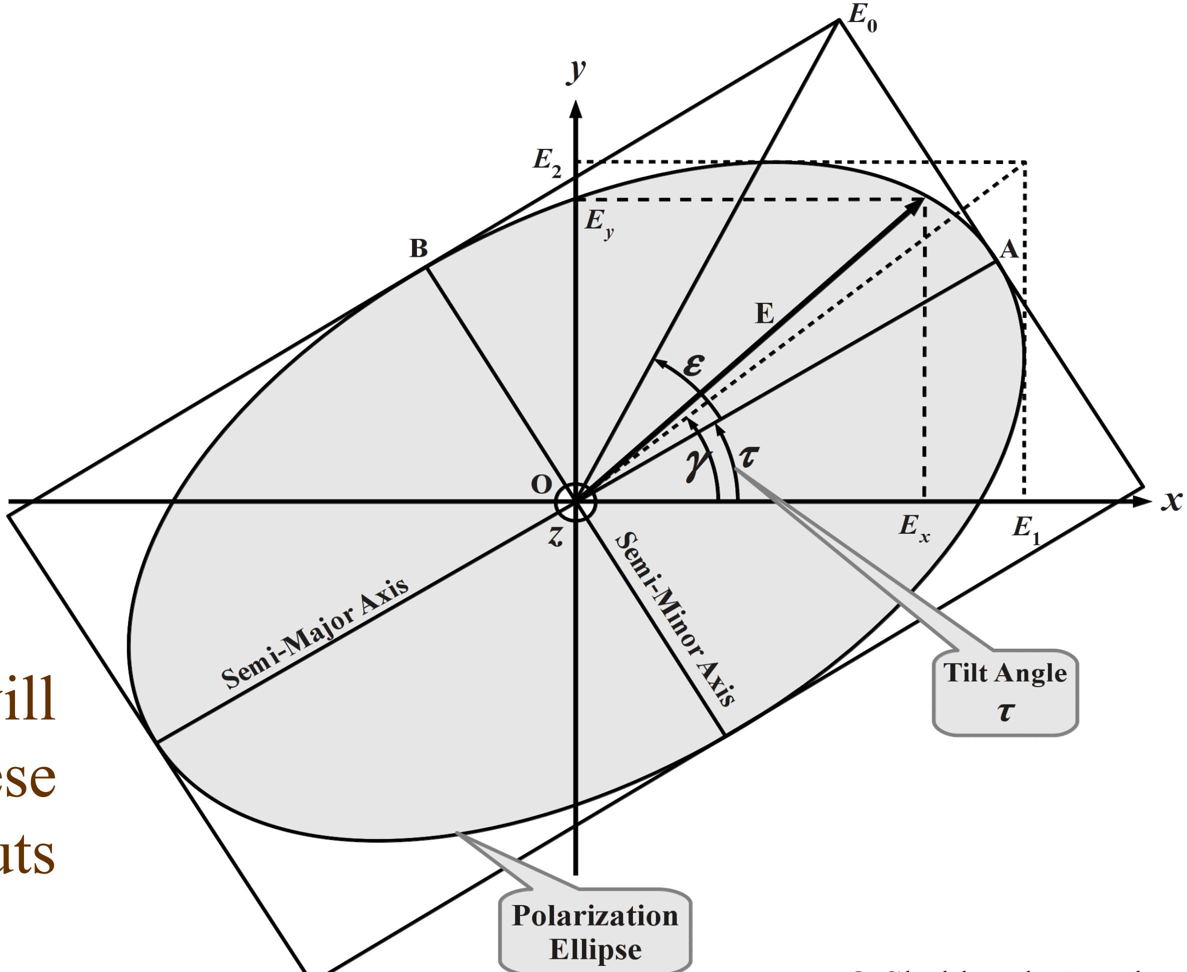
Stokes Pars: Fully polarized waves

We now investigate the Stokes parameters for a ***completely polarized wave*** using a pair of dipoles which are crossed. The orientation of the red dipole is along x-axis. The blue dipole is oriented along y-axis.





The Stokes parameters will be formed from these linearly polarized outputs of the dipoles.

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Stokes Pars: Fully polarized waves

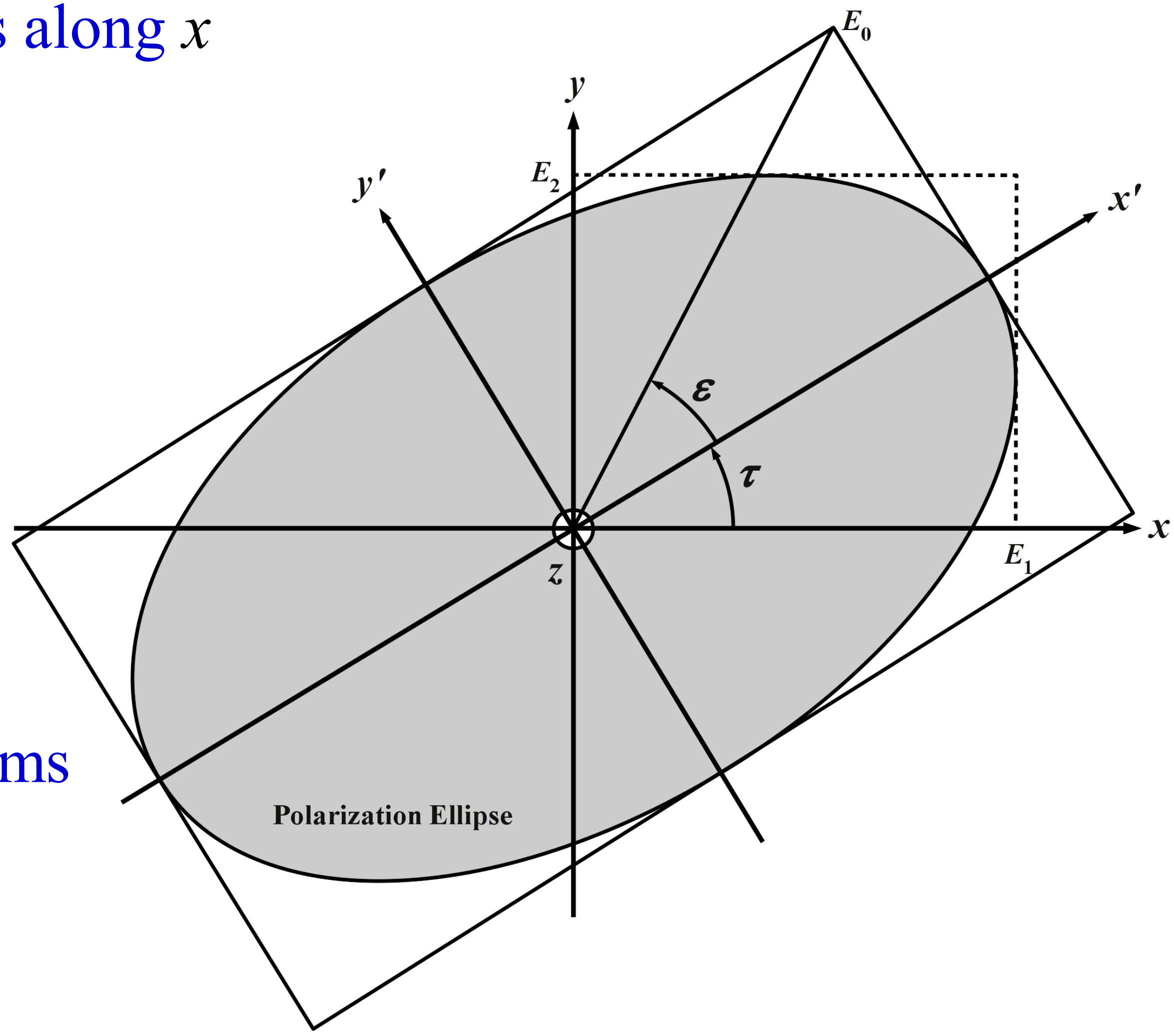
At $z = 0$, the electric fields along x and y are:

$$E_x = E_1 \sin(\omega t - \delta_1)$$
$$E_y = E_2 \sin(\omega t - \delta_2)$$

Phase difference = $\delta_1 - \delta_2$

Electric field components (elliptical) along x' and y':

$$E'_x = E_0 \cos\epsilon \, \sin\omega t$$
$$E'_y = E_0 \sin\epsilon \, \cos\omega t$$



Expressing E_x and E_y in terms of E'_x and E'_y we get:

$$E_x = E'_x \cos\tau - E'_y \sin\tau$$
$$E_y = E'_x \sin\tau + E'_y \cos\tau$$

Some algebraic manipulation gives:

$$E_x = E_0[\cos\epsilon\cos\tau\sin\omega t - \sin\epsilon\sin\tau\cos\omega t]$$
$$E_y = E_0[\cos\epsilon\sin\tau\sin\omega t - \sin\epsilon\cos\tau\cos\omega t]$$

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Stokes Pars: Fully polarized waves

We have, $E_x = E_1 \sin(\omega t - \delta_1)$

or, $E_x = (E_1 \cos\delta_1) \sin\omega t - (E_1 \sin\delta_1) \cos\omega t$

Also we have $E_x = (E_0 \cos\epsilon \cos\tau) \sin\omega t - (E_0 \sin\epsilon \sin\tau) \cos\omega t$

Equating the terms containing sinωt we get $E_1 \cos\delta_1 = E_0 \cos\epsilon \cos\tau$

Equating the terms containing cosωt we get $E_1 \sin\delta_1 = E_0 \sin\epsilon \sin\tau$

Squaring the two terms and adding we eliminate sinδ_1 and cosδ_1

Thus we get $E_1 = E_0 \sqrt{\cos^2\epsilon\ \cos^2\tau + \sin^2\epsilon\ \sin^2\tau}$

Similarly using the other two equations:

$$E_y = E_2 \sin(\omega t - \delta_2)$$

$$E_y = E_0 [\cos\epsilon \sin\tau \sin\omega t - \sin\epsilon \cos\tau \cos\omega t]$$

we get $E_2 = E_0 \sqrt{\cos^2\epsilon\ \sin^2\tau + \sin^2\epsilon\ \cos^2\tau}$

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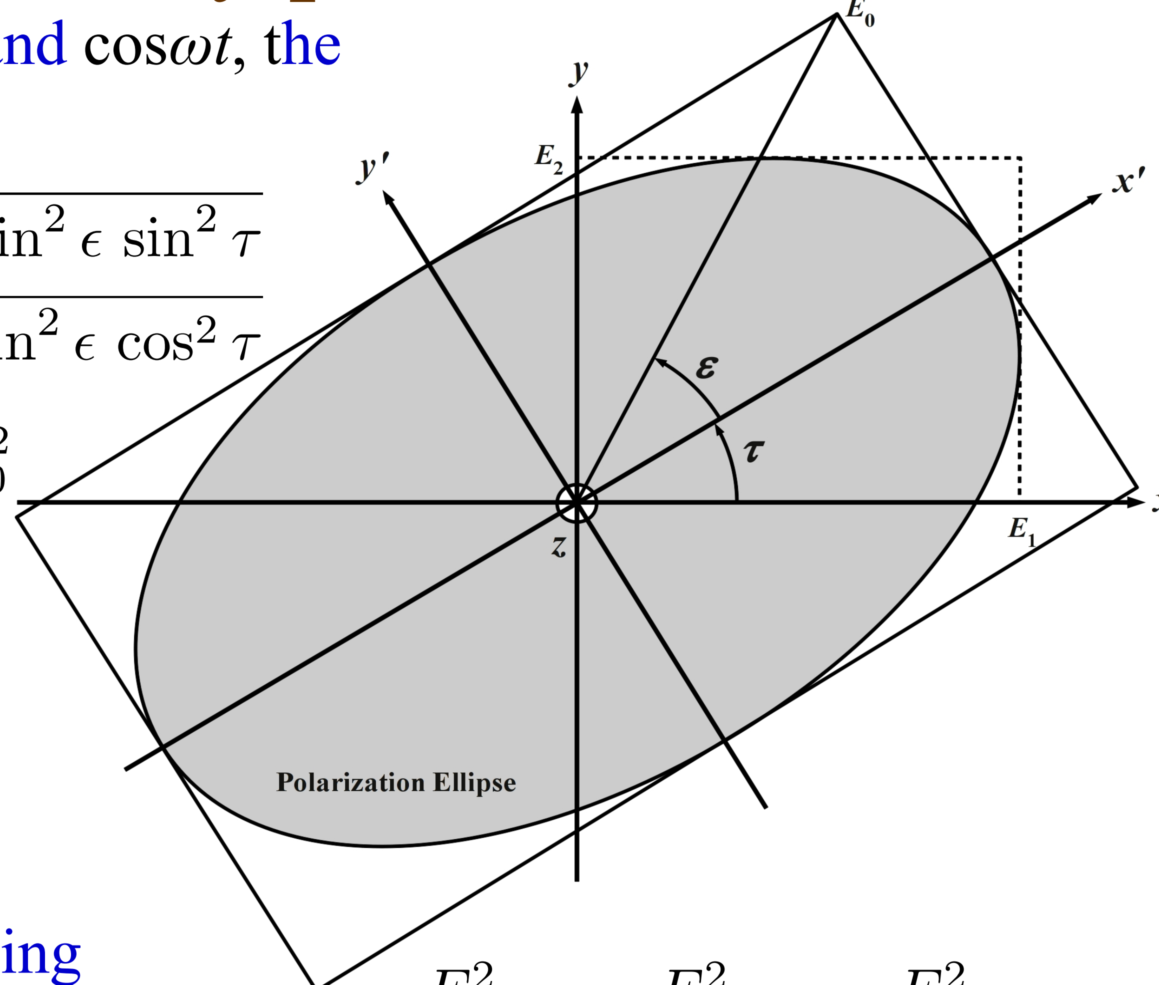
Stokes Pars: Fully polarized waves

Having eliminated sinωt and cosωt, the values of E_1 and E_2 are:

$$E_1 = E_0 \sqrt{\cos^2 \epsilon \, \cos^2 \tau + \sin^2 \epsilon \, \sin^2 \tau}$$

$$E_2 = E_0 \sqrt{\cos^2 \epsilon \, \sin^2 \tau + \sin^2 \epsilon \, \cos^2 \tau}$$

Note that: $E_1^2 + E_2^2 = E_0^2$



The effective values are:

$$E_{1eff} = E_1/\sqrt{2}$$

$$E_{2eff} = E_2/\sqrt{2}$$

$$E_{0eff} = E_0/\sqrt{2}$$

Magnitude of total Poynting vector **S** (watts/m^2) is:

$$S = S_x + S_y = \frac{E_{1eff}^2}{Z_0} + \frac{E_{2eff}^2}{Z_0} = \frac{E_{0eff}^2}{Z_0}$$

where, Z (ohms/square) is the impedance of medium. S_x and S_y are polarized wave components of S along x and y directions.

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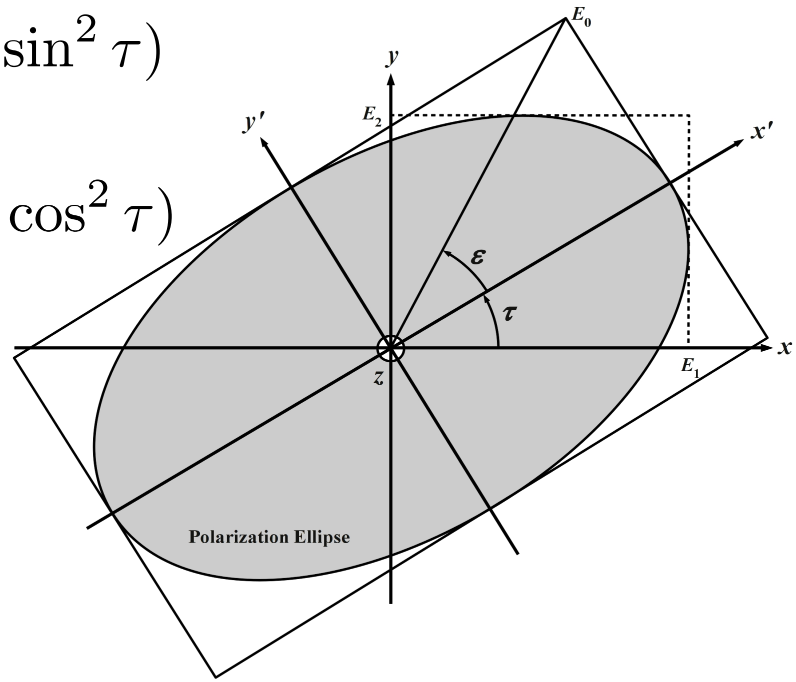
Stokes Pars: Fully polarized waves

$$S = S_x + S_y = \frac{E_{1eff}^2}{Z_0} + \frac{E_{2eff}^2}{Z_0} = \frac{E_{0eff}^2}{Z_0}$$

where,

$$S_x = \frac{E_{1eff}^2}{Z_0} = S(\cos^2\epsilon\,\cos^2\tau + \sin^2\epsilon\,\sin^2\tau)$$

$$S_y = \frac{E_{2eff}^2}{Z_0} = S(\cos^2\epsilon\,\sin^2\tau + \sin^2\epsilon\,\cos^2\tau)$$



Stokes params. of completely polarized waves using linear components are:

$$I = S = S_x + S_y = \frac{E_{1eff}^2}{Z_0} + \frac{E_{2eff}^2}{Z_0}$$

$$Q = S_x - S_y = \frac{E_{1eff}^2}{Z_0} - \frac{E_{2eff}^2}{Z_0} = S\,\cos 2\epsilon\,\cos 2\tau$$

$$U = (S_x - S_y)\tan 2\tau = S\,\cos 2\epsilon\,\sin 2\tau = 2\frac{E_{1eff}\,E_{2eff}}{Z_0}\cos(\delta_1 - \delta_2)$$

$$V = (S_x - S_y)\tan 2\epsilon\,\sec 2\tau = S\,\sin 2\epsilon = 2\frac{E_{1eff}\,E_{2eff}}{Z_0}\sin(\delta_1 - \delta_2)$$

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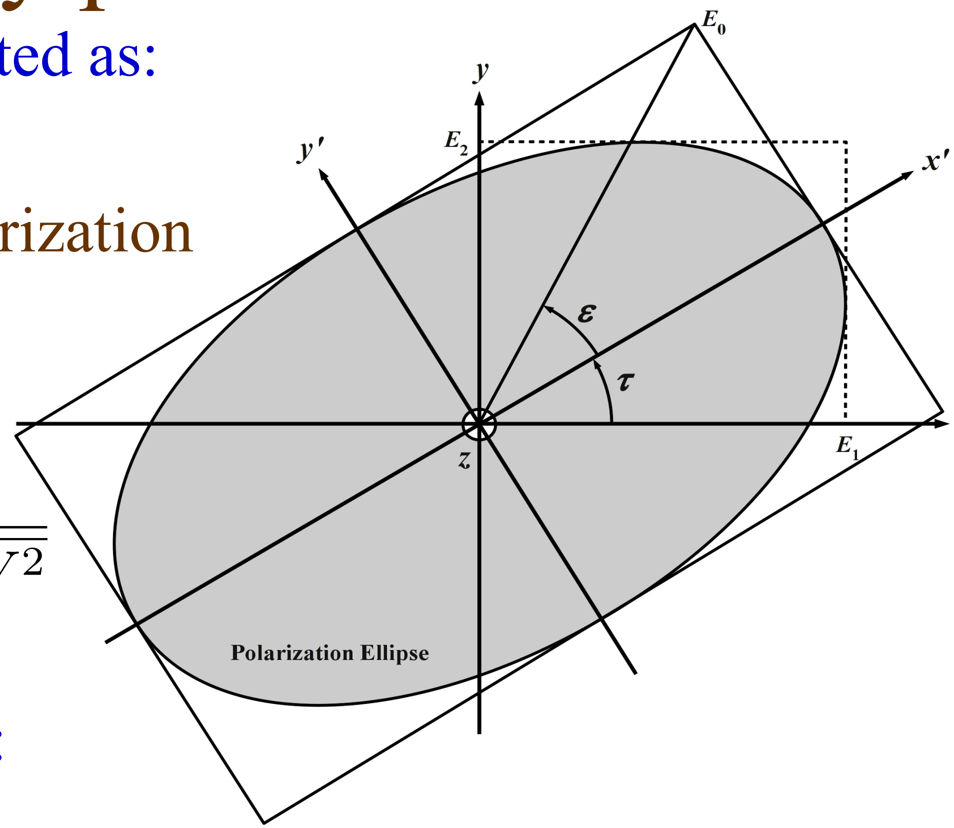
Stokes Pars: Fully polarized waves

The four Stokes parameters are related as:

$$I = \sqrt{Q^2 + U^2 + V^2}$$

The angular relations w.r.t. the polarization ellipse: $\dfrac{U}{Q} = \tan 2\tau$

$$\frac{V}{S} = \sin 2\epsilon = \frac{V}{\sqrt{Q^2 + U^2 + V^2}}$$



Stokes parameters for completely polarized waves in a matrix form is:

$$\begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} = \begin{bmatrix} S_x + S_y \\ S_x - S_y \\ (S_x - S_y)\tan 2\tau \\ (S_x - S_y)\tan 2\epsilon\ \sec 2\tau \end{bmatrix} = \begin{bmatrix} S \\ S\ \cos 2\epsilon\ \cos 2\tau \\ 2\frac{E_{1eff}\ E_{2eff}}{Z_0}\cos(\delta_1 - \delta_2) \\ 2\frac{E_{1eff}\ E_{2eff}}{Z_0}\sin(\delta_1 - \delta_2) \end{bmatrix}$$

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Stokes Parameters

Stokes parameters for completely polarized waves using a dipole cross.

Stokes Parameters	Left Circular	Right Circular	Linear Horizontal	Linear Vertical	Linear +45°	Linear –45°
I	*S*	*S*	*S*	*S*	*S*	*S*
Q	0	0	*S*	-*S*	0	0
U	0	0	0	0	*S*	-*S*
V	*S*	-*S*	0	0	0	0

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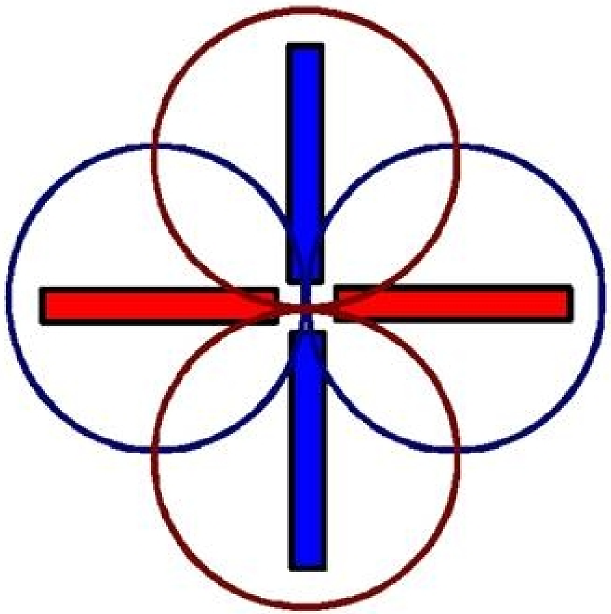
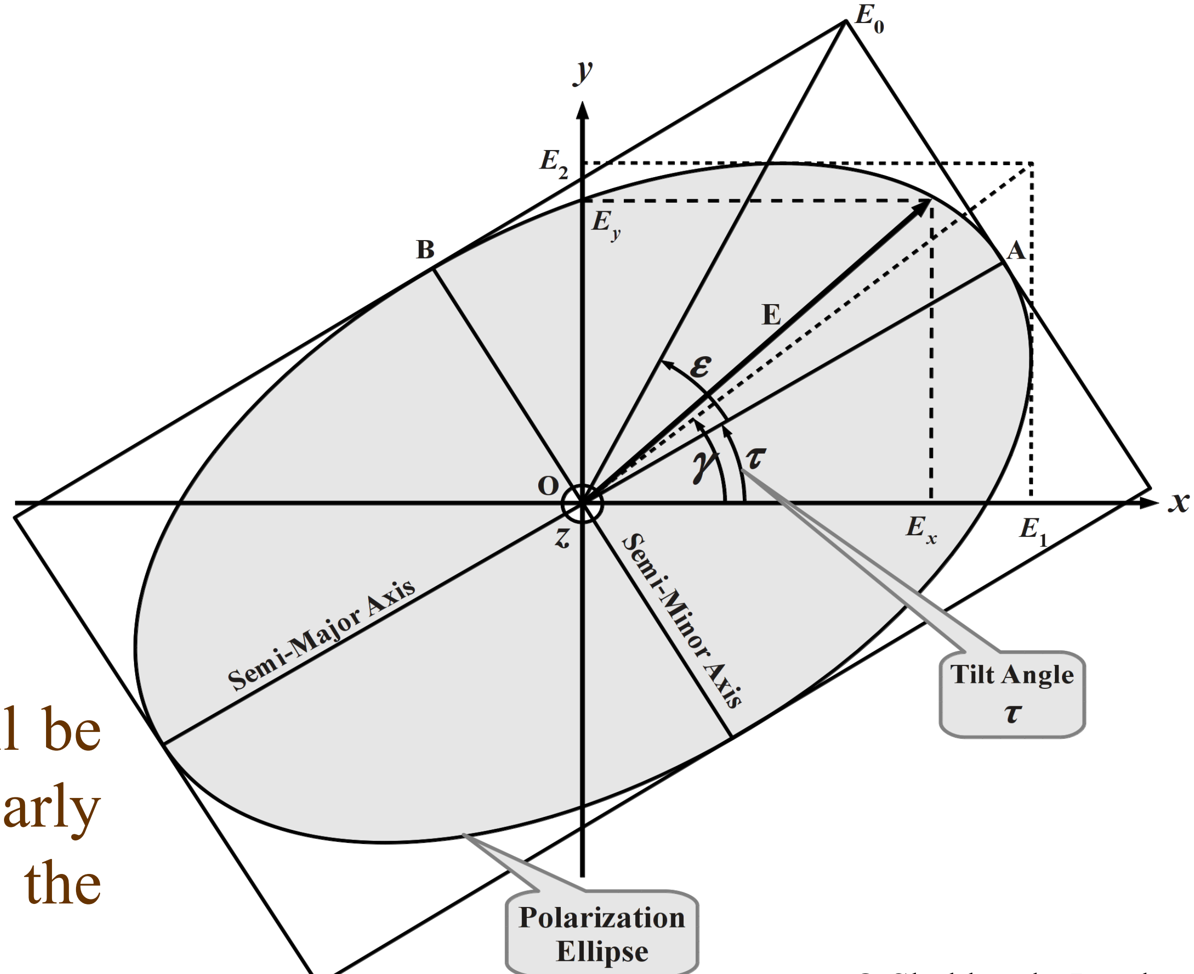
Stokes Pars: Fully un-polarized waves

We now investigate the Stokes parameters for a ***completely un-polarized wave*** using two dipoles forming a cross. The orientation of the red dipole is along *x*-axis. The blue dipole is oriented along *y*-axis.





The Stokes parameters will be formed from these linearly polarized outputs of the dipoles.

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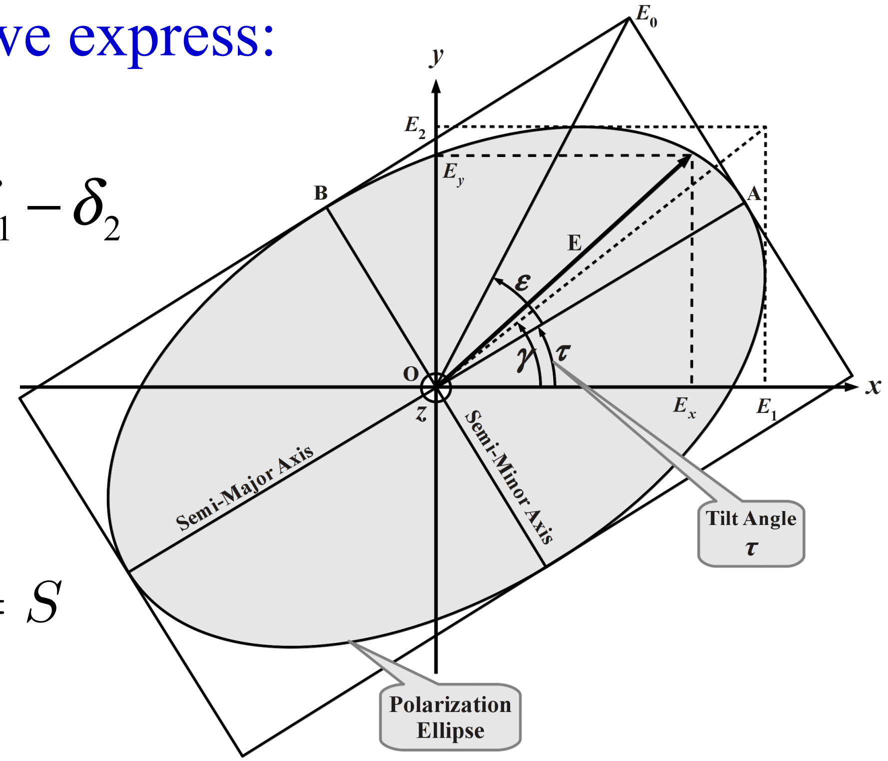
Stokes Pars: Fully un-polarized waves

For a completely un-polarized wave we express:

$$E_x = E_1(t)\sin(\omega t - \delta_1(t))$$

$$E_y = E_2(t)\sin(\omega t - \delta_2(t))$$ and $\delta = \delta_1 - \delta_2$

Since δ_1 and δ_2 varies randomly, we consider the mean variations over time. The Stokes parameters are:



$$I = \frac{\left\langle E^2_{1eff}\right\rangle}{Z_0} + \frac{\left\langle E^2_{2eff}\right\rangle}{Z_0} = S_x + S_y = S$$

$$Q = \frac{\left\langle E^2_{1eff}\right\rangle}{Z_0} - \frac{\left\langle E^2_{2eff}\right\rangle}{Z_0} = S_x - S_y = S\left\langle \cos 2\varepsilon\ \cos 2\tau\right\rangle$$

$$U = 2\frac{\langle E_{1eff}\, E_{2eff}\cos\delta\rangle}{Z_0} = S\left\langle \cos 2\varepsilon\ \sin 2\tau\right\rangle$$

$$V = 2\frac{\langle E_{1eff}\, E_{2eff}\ \sin\delta\rangle}{Z_0} = S\left\langle \sin 2\varepsilon\ \right\rangle$$

where, $\langle E^2_{1eff}\rangle = \frac{1}{T}\int_0^T [E_{1eff}(t)]^2\, dt$ and $\langle E^2_{2eff}\rangle = \frac{1}{T}\int_0^T [E_{2eff}(t)]^2\, dt$

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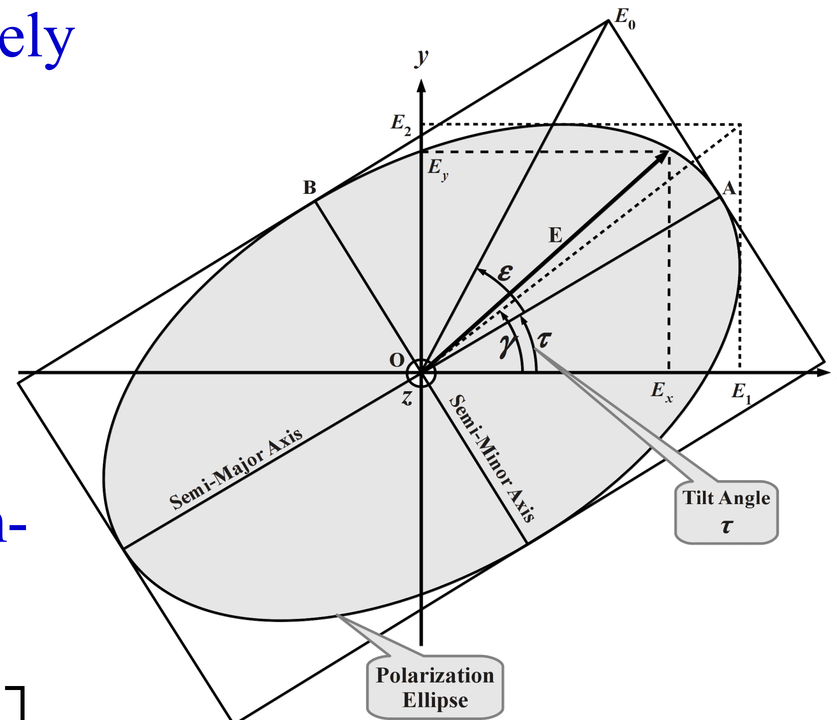
Stokes Pars: Fully un-polarized waves

The Stokes parameters for completely un-polarized waves are related as:

$$I \geq \sqrt{Q^2 + U^2 + V^2}$$

Note that the *sum-equality may or may not hold here.*



Stokes parameters for completely un-polarized waves in a matrix form is:

$$\begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} = \frac{1}{Z_0} \begin{bmatrix} \left\langle E_{1eff}^2 \right\rangle + \left\langle E_{2eff}^2 \right\rangle \\ \left\langle E_{1eff}^2 \right\rangle - \left\langle E_{2eff}^2 \right\rangle \\ 2 \left\langle E_{1eff} E_{2eff} \cos\delta \right\rangle \\ 2 \left\langle E_{1eff} E_{2eff} \sin\delta \right\rangle \end{bmatrix} = \begin{bmatrix} S \\ S \left\langle \cos 2\epsilon \, \cos 2\tau \right\rangle \\ S \left\langle \cos 2\varepsilon \, \sin 2\tau \right\rangle \\ S \left\langle \sin 2\varepsilon \right\rangle \end{bmatrix}$$

Degree of polarization $d = \dfrac{\text{polarized power}}{\text{total power}} = \dfrac{\sqrt{Q^2 + U^2 + V^2}}{I}, \quad 0 \leq d \leq 1$

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Stokes Pars: Fully un-polarized waves

Summary of Stokes parameters for completely un-polarized waves

(*i*) I is power from sum of autocorrelations of x and y components.
(*ii*) Q is the difference of autocorrelations of x and y components.
(*iii*) U is a cross-correlation function of E_1 with $E_2 \cos\delta$.
(iv) V is a cross-correlation function of E_1 with $E_2 \sin\delta$.

$$\boldsymbol{I \sim R(E_1) + R(E_2)}$$

$$\boldsymbol{U \sim r(E_1, E_2 \cos\delta)}$$

$$\boldsymbol{Q \sim R(E_1) - R(E_2)}$$

$$\boldsymbol{V \sim r(E_1, E_2 \sin\delta)}$$

FUNCTIONS: ***R*** **-** auto-correlation, ***r*** **-** cross-correlation

Note: For completely un-polarized waves, there is no correlation between E_{1eff} and E_{2eff} though their magnitudes are same ($S_x = S_y$). Therefore, $\langle E_{1eff}, E_{2eff} \cos\delta \rangle = 0$. Similarly, $\langle E_{1eff}, E_{2eff} \sin\delta \rangle = 0$.

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Stokes Parameters

Stokes parameters for completely un-polarized waves using a dipole cross.

Stokes parameters	**Values for completely un-polarized waves**
$\boldsymbol{I}$	S
$\boldsymbol{Q}$	0
$\boldsymbol{U}$	0
$\boldsymbol{V}$	0

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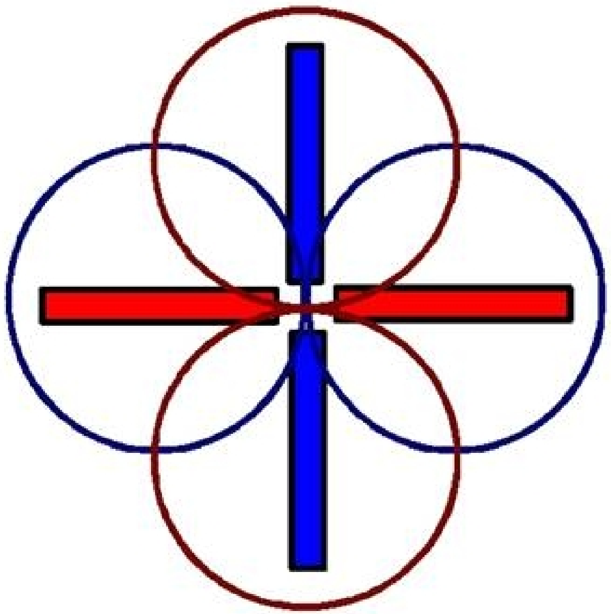
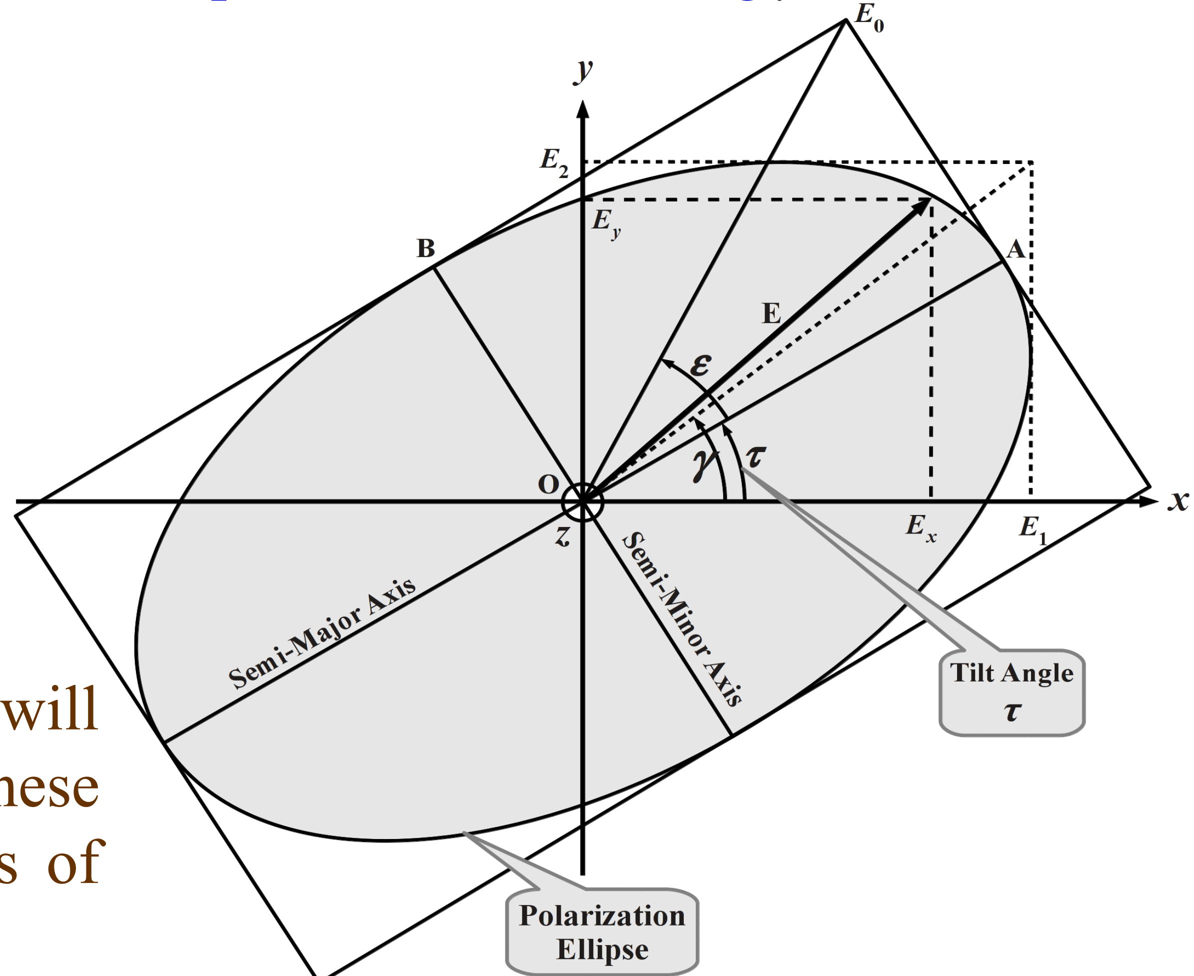
Stokes Pars: Partially polarized waves

We now investigate the Stokes parameters for a ***partially polarized wave*** using two dipoles forming a cross. The orientation of the red dipole is along *x*-axis. The blue dipole is oriented along *y*-axis.





The Stokes parameters will be formed from these linearly polarized outputs of the dipoles.

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Stokes Pars: Partially polarized waves

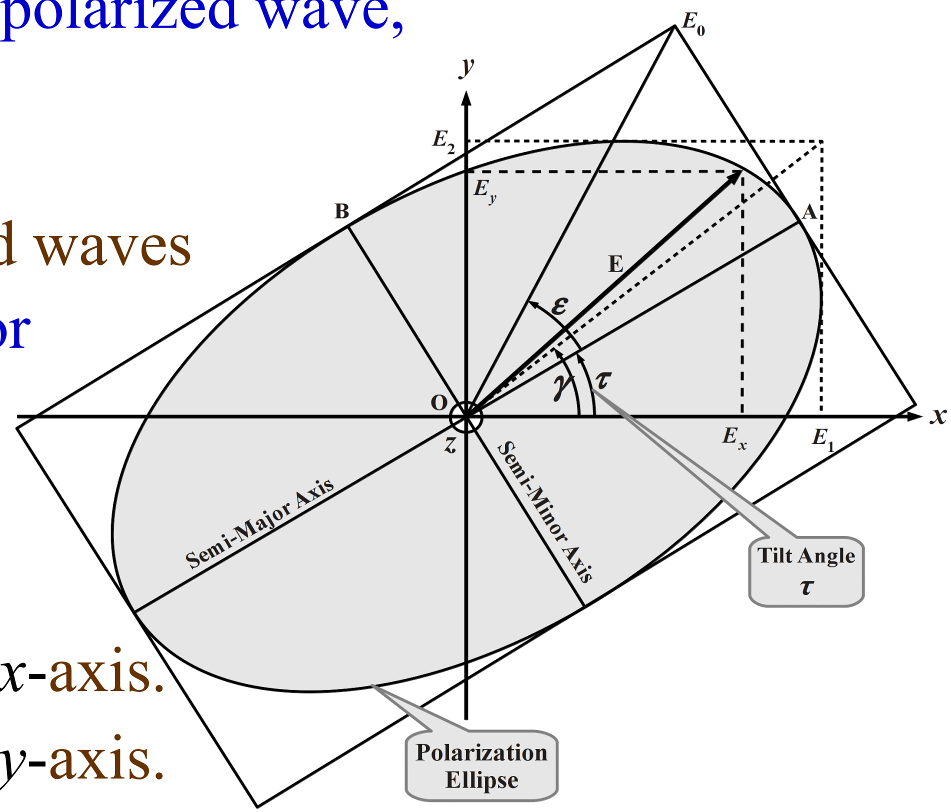
We have seen, for a completely un-polarized wave,

$$Q = U = V = 0.$$

Partially polarized wave

= Polarized waves + Un-polarized waves

Hence, at least one among Q or U or V is non-zero.



Let the flux densities be:

S_u - Un-polarized components.

S_{xp} - Components polarized along x-axis.

S_{yp} - Components polarized along y-axis.

Thus the Stokes parameters of a partially polarized wave are:

$$I = S = S_u + S_{xp} + S_{yp} \qquad Q = S_{xp} - S_{yp}$$

$$U = (S_{xp} - S_{yp}) \tan 2\tau \qquad V = (S_{xp} - S_{yp}) \tan 2\varepsilon \ \sec 2\tau$$

It is a generic equation which can describe any type of polarization.

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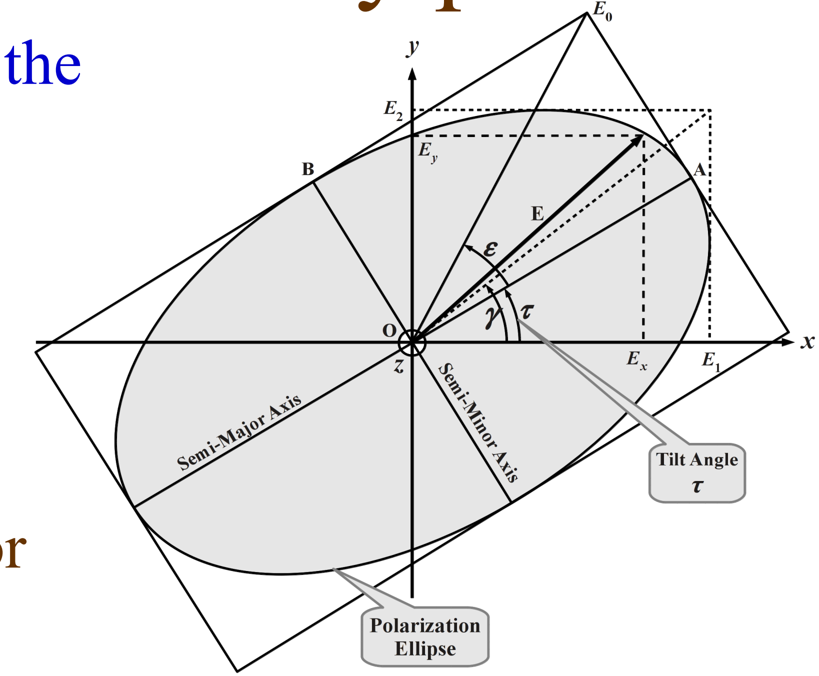
Normalized Stokes Pars: Partially polarized

The Stokes parameters are also used in the normalized form:



$$s_0 = \frac{I}{S} = 1 \qquad s_1 = \frac{Q}{S}$$

$$s_2 = \frac{U}{S} \qquad s_3 = \frac{V}{S}$$

The Stokes parameters in matrix form for partially polarized waves is:

$$\begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} = S \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = S \begin{bmatrix} 1-d \\ 0 \\ 0 \\ 0 \end{bmatrix} + S \begin{bmatrix} d \\ d\cos 2\varepsilon \, \cos 2\tau \\ d\cos 2\varepsilon \, \sin 2\tau \\ d\sin 2\varepsilon \end{bmatrix}$$

Thus, partially polarized = Completely polarized + Un-polarized

Degree of polarization $d = \dfrac{\sqrt{Q^2+U^2+V^2}}{I} = \sqrt{s_1^2+s_2^2+s_3^2}$

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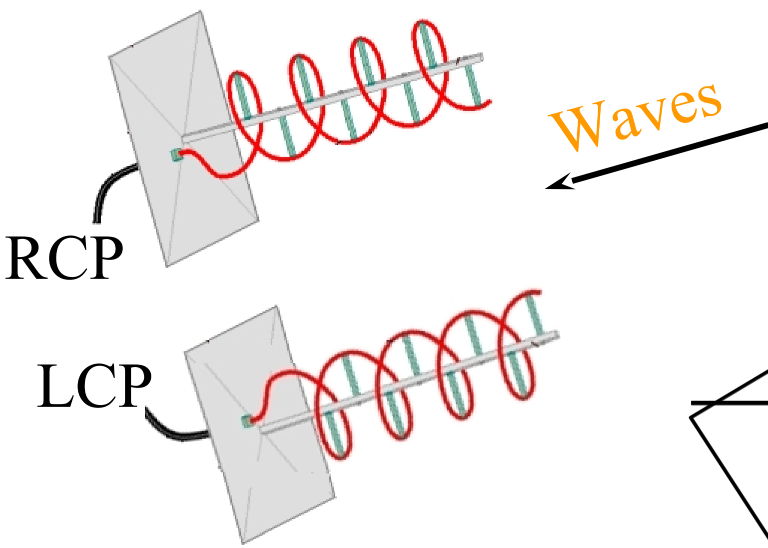
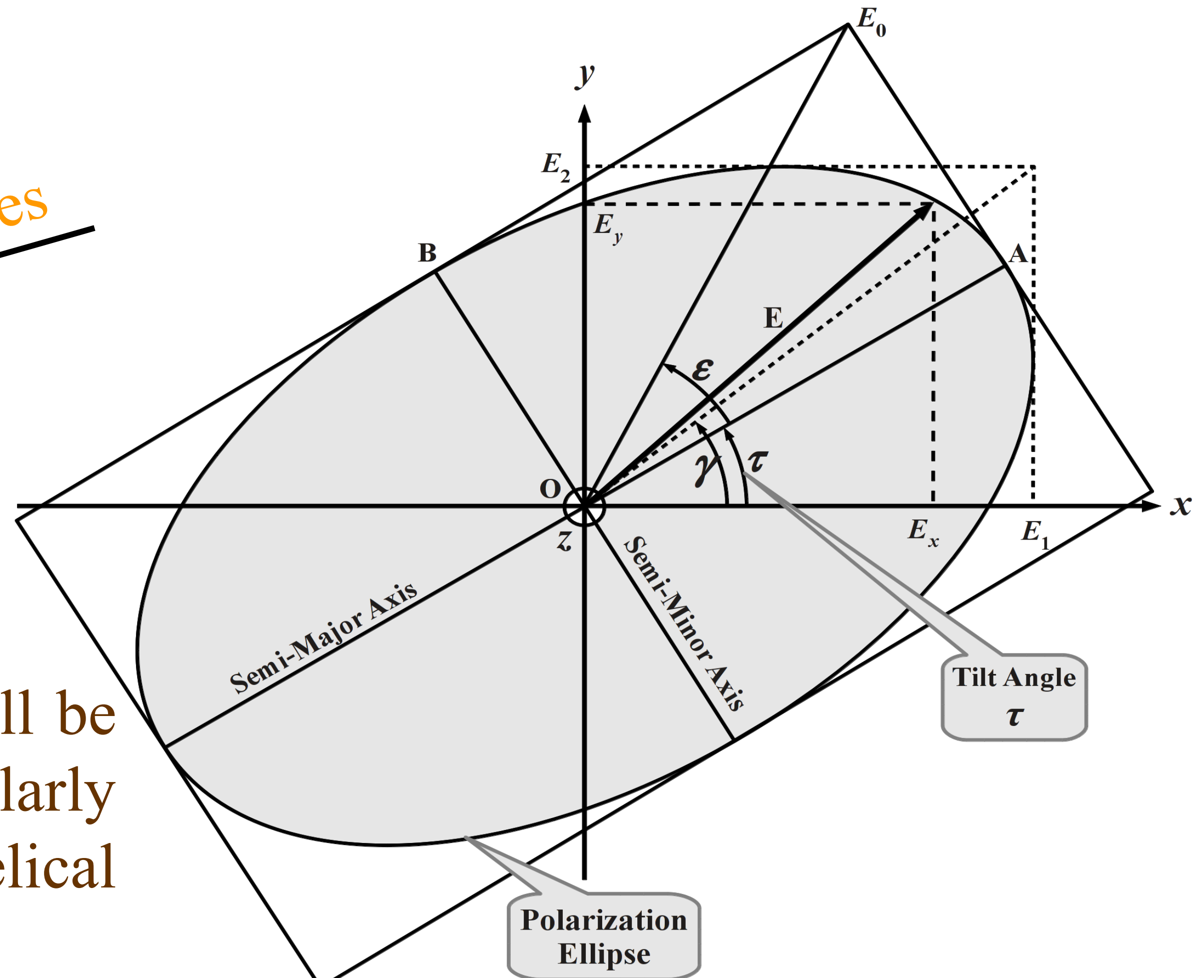
Stokes Pars: Alternative Approach

We now investigate the Stokes parameters for a ***partially polarized wave*** using two oppositely polarized helical antennas, producing right and left circularly polarized waves.





The Stokes parameters will be formed from these circularly polarized outputs of the helical antennas.

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Stokes Pars: Alternative Approach

RCP $E_r = E_R\, e^{j\omega t}$

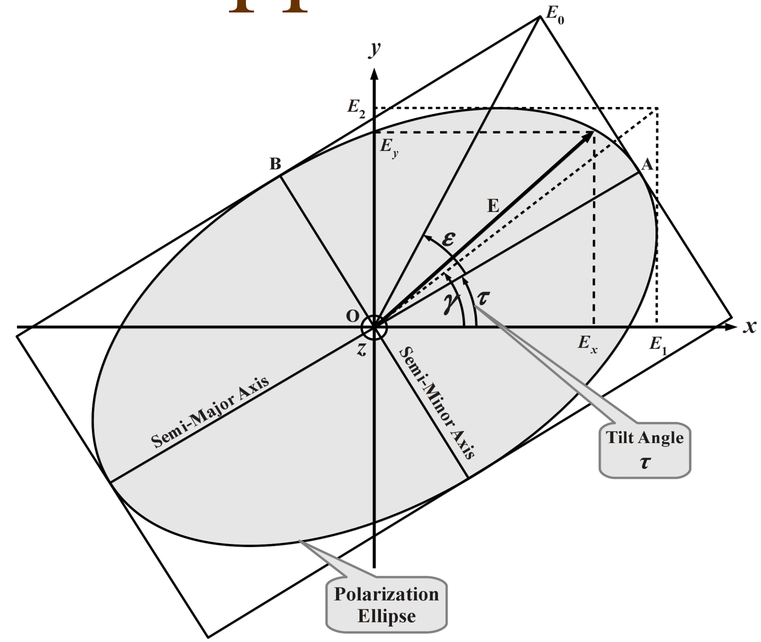
LCP $E_l = E_L\, e^{-j(\omega t + \delta')}$

Axial Ratio $AR = \dfrac{E_{Leff} + E_{Reff}}{E_{Leff} - E_{Reff}} = \cot \epsilon$

where, E_{Leff} and E_{Reff} are the effective values of E_L and E_R respectively.

If $E_L > E_R$, AR is positive (LCP).

If $E_L < E_R$, AR is negative (RCP).

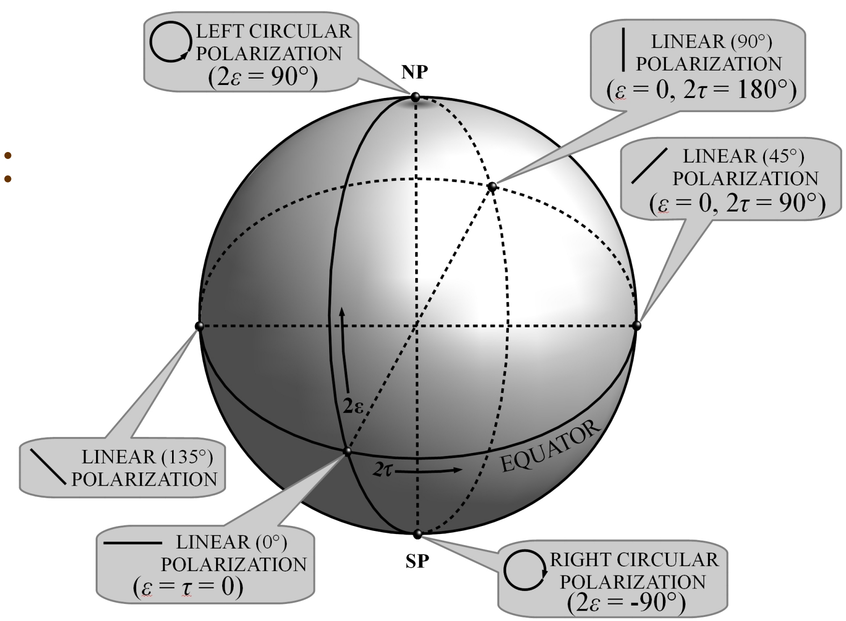


Relating with Poincaré sphere we get:

$$\cos 2\epsilon = \frac{2E_{Leff}\, E_{Reff}}{E_{Leff}^2 + E_{Reff}^2} = \frac{(AR)^2 - 1}{(AR)^2 + 1}$$

$$\sin 2\epsilon = \frac{E_{Leff}^2 - E_{Reff}^2}{E_{Leff}^2 + E_{Reff}^2} = \frac{2\,(AR)}{(AR)^2 + 1}$$

$$\delta' = 2\tau$$



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Stokes Pars: Alternative Approach

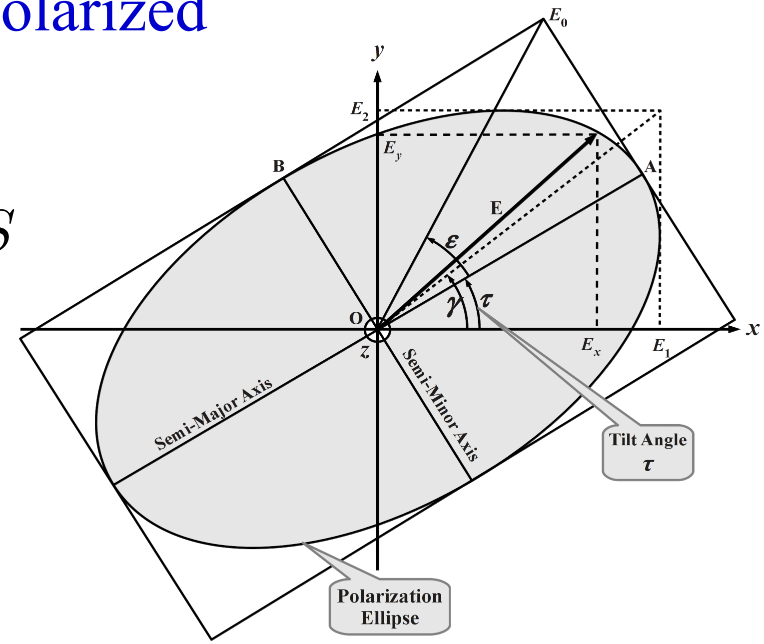
Stokes parameters in terms of circularly polarized components are:

$$I = \frac{\left\langle E^2_{Leff} \right\rangle}{Z_0} + \frac{\left\langle E^2_{Reff} \right\rangle}{Z_0} = S_L + S_R = S$$

$$Q = \frac{2}{Z_0} \langle E_{Leff}\, E_{Reff} \cos\delta' \rangle$$

$$U = \frac{2}{Z_0} \langle E_{Leff}\, E_{Reff} \sin\delta' \rangle$$

$$V = \frac{\left\langle E^2_{Leff} \right\rangle}{Z_0} - \frac{\left\langle E^2_{Reff} \right\rangle}{Z_0} = S_L - S_R$$

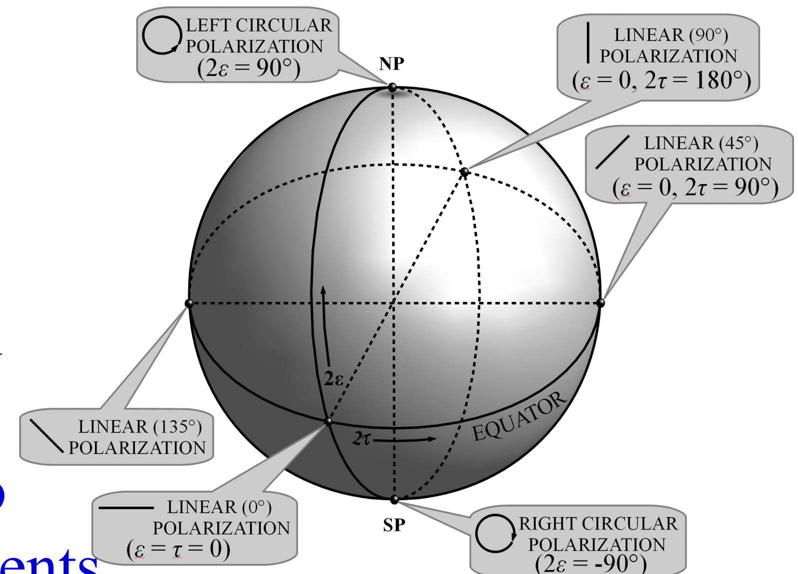


Interpretation of Stokes parameters:

(*i*) ***I*** is the power sum of RCP and LCP.

(*ii*) ***V*** is the power difference of RCP and LCP.

(*iii*) ***Q*** and ***U*** are cross-correlations of two opposite circularly polarized components.



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Stokes Pars: Alternative Approach

The Stokes parameters which are expressed in terms of circularly polarized components can be now written in a matrix form as shown below:

$$\begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} = \begin{bmatrix} \frac{\left\langle E_{Leff}^2 \right\rangle}{Z_0} + \frac{\left\langle E_{Reff}^2 \right\rangle}{Z_0} \\ 2\frac{\left\langle E_{Leff}\, E_{Reff} \cos\delta' \right\rangle}{Z_0} \\ 2\frac{\left\langle E_{Leff}\, E_{Reff}\ \sin\delta' \right\rangle}{Z_0} \\ \frac{\left\langle E_{Leff}^2 \right\rangle}{Z_0} - \frac{\left\langle E_{Reff}^2 \right\rangle}{Z_0} \end{bmatrix} = \begin{bmatrix} S_L + S_R \\ 2\frac{\langle E_{Leff}\, E_{Reff} \cos 2\tau \rangle}{Z_0} \\ 2\frac{\langle E_{Leff}\, E_{Reff}\ \sin 2\tau \rangle}{Z_0} \\ S_L - S_R \end{bmatrix}$$

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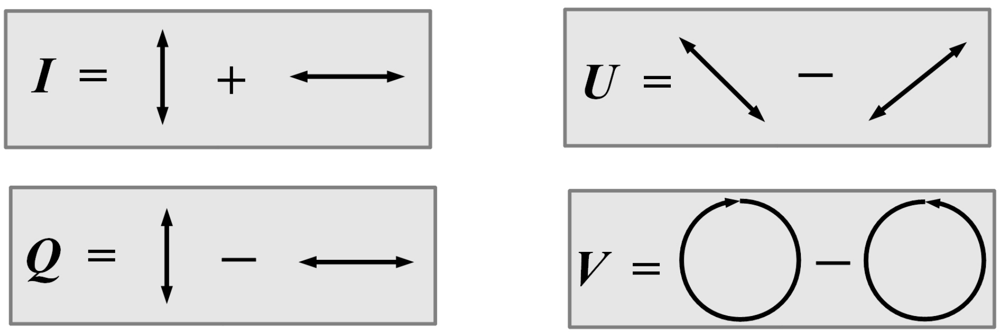
Stokes Parameters: Final Resolution

We now compare the Stokes parameters obtained from (*i*) linearly polarized antenna pair with (*ii*) those from the circularly polarized antenna pair. Based on these the final model of the Stokes parameters is shown in the figure, and in matrix form.



$$\begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} = \begin{bmatrix} S_{0^\circ} + S_{90^\circ} \\ S_{0^\circ} - S_{90^\circ} \\ S_{45^\circ} - S_{135^\circ} \\ S_{\mathrm{RCP}} - S_{\mathrm{LCP}} \end{bmatrix}$$

Thus, the Stokes parameters can be thought as the combination of power components obtained from (*i*) linear crossed dipoles at 0º and 90º, (*ii*) linear crossed dipoles at 45º and 135º, and (*iii*) two oppositely polarized helical antennas.

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Stokes Parameters and Antenna Aperture

We relate the Stokes parameters in a matrix form with the aperture of a antenna pair receiving arbitrary polarized waves. Assume that both antennas in the pair have same effective aperture area A_e. If they are linearly polarized, assume their polarization directions as crossed (perpendicular to each other). If they are circularly polarized, assume the two to be oppositely polarized (right and left circular). The effective aperture area A_e of the pair have four components, which are expressed in a matrix form as:

$$A_e [a_i] = A_e \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

where, a_i represents the normalized Stokes parameters.

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Stokes Pars: Antenna in Tranmitting Mode

When the antennas are transmitting, a_i is given by:

$$a_0 = 1$$

$$a_1 = \frac{1}{Z\,S_t}\left(E_{1t}^2 - E_{2t}^2\right)$$

$$a_2 = \frac{2}{Z\,S_t}\left(E_{1t}\,E_{2t}\,\cos\delta\right)$$

$$a_3 = \frac{2}{Z\,S_t}\left(E_{1t}\,E_{2t}\,\sin\delta\right)$$

where, S_t is the transmitted power, E_{1t} and E_{2t} are respectively the effective values of the electric fields and δ is the phase difference between them.

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Stokes Pars: Antenna in Receiving Mode

When the antennas are receiving, and a wave of polarization $S[s_i]$ is incident, the power W available from the antennas is given as:

$$W = \frac{1}{2} S\, A_e\, [a_i]^T\, [s_i]$$

$$= \frac{1}{2} S\, A_e \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

$$= \frac{1}{2} S\, A_e \sum_{i=0}^{3} a_i\, s_i$$

Here, s_i is the normalized Stokes parameters of the incident waves.

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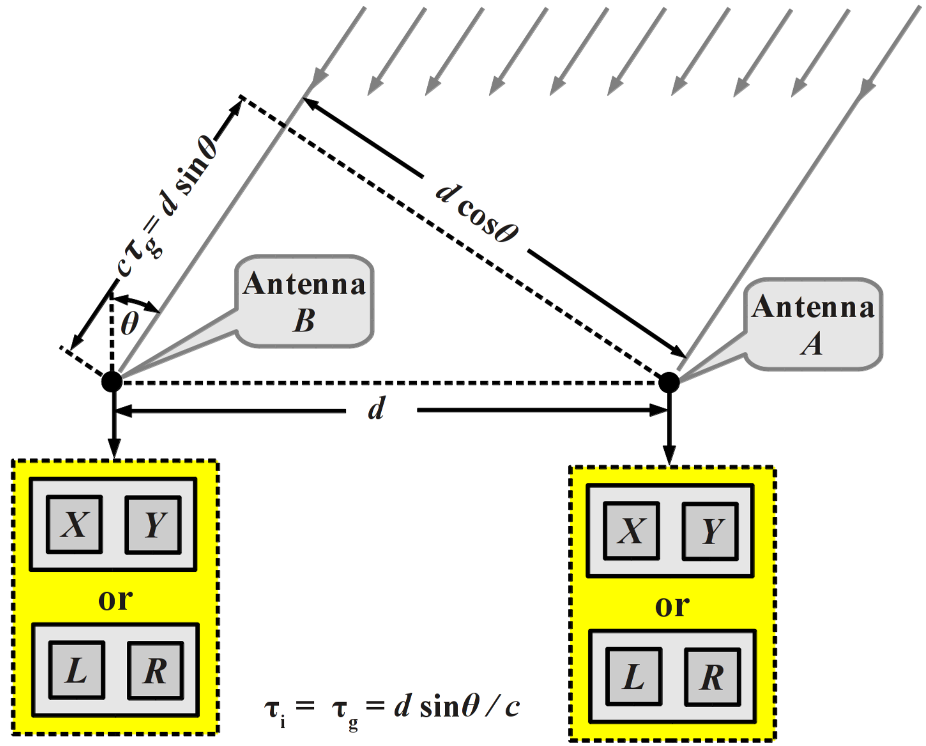
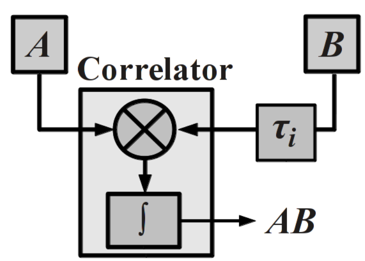
Stokes Parameters for Interferometers

Two antennas separated by a distance *d* and tracking a radio source whose outputs are multiplied in phase followed by time-integration is an interferometer. Since each antenna receives two polarizations (linear or circular), four correlated outputs (two self and two cross) are produced. Antenna gains for all polarizations are assumed identical.

Note: We are considering only cross products. Auto-correlations are not used.





Linear Polarization

A	*B*	Output
X	*X*	*XX*
X	*Y*	*XY*
Y	*X*	*YX*
Y	*Y*	*YY*

Circular Polarization

A	*B*	Output
L	*L*	*LL*
L	*R*	*LR*
R	*L*	*RL*
R	*R*	*RR*

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Stokes Parameters: Linearly Polarized Interferometers

For linearly polarized reception, the correlations are related to Stokes parameters as:

$$XX\left[\frac{1}{Z_{TL}\,A_e}\right] = I + Q \qquad XY\left[\frac{1}{Z_{TL}\,A_e}\right] = U + jV$$

$$YX\left[\frac{1}{Z_{TL}\,A_e}\right] = U - jV \qquad YY\left[\frac{1}{Z_{TL}\,A_e}\right] = I - Q$$

where, X and Y are the two polarized output voltages from each antenna in the equatorial reference frame of the source, Z_{TL} is the characteristic impedance of the transmission line matched to the antenna, and A_e is the effective aperture area of each antenna.

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Stokes Parameters: Circularly Polarized Interferometers

For circularly polarized reception, the correlations are related to Stokes parameters as:

$$LL\left[\frac{1}{Z_{TL}\,A_e}\right] = I - V \qquad LR\left[\frac{1}{Z_{TL}\,A_e}\right] = Q - jU$$

$$RL\left[\frac{1}{Z_{TL}\,A_e}\right] = Q + jU \qquad RR\left[\frac{1}{Z_{TL}\,A_e}\right] = I + V$$

where, L and R are the two polarized output voltages from each antenna in the equatorial reference frame of the source, Z_{TL} is the characteristic impedance of the transmission line matched to the antenna, and A_e is the effective aperture area of each antenna.

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Stokes Parameters for Interferometers...

Important Note

In the equations we have shown for interferometer, we have assumed the antennas as ideal. Practically, antenna-feeds are not perfectly polarized. Hence an unpolarized source appears as polarized when observed using them. Furthermore, while tracking the source, the antenna-feeds rotate relative to the equatorial frame of the source. Thus, the above equations have to be modified accordingly before usage. A polarization calibration is generally done to remove both these effects. After calibration, only the source polarization information remains in the data.

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Assignment Problems-I

1. What is meant by polarization of an electromagnetic wave? Define the terms (*i*) linear polarization, (*ii*) circular polarization, and (*iii*) elliptical polarization.

2. What is mean by unpolarized wave?

3. Using a diagram explain the polarization ellipse.

4. If E_1 and E_2 are the maximum magnitudes of x and y components of the electric field, the angular frequency being ω, δ is the time phase angle by which E_y leads over E_x, and $\mathbf{e_x}$ and $\mathbf{e_y}$ are respectively the unit vectors along x and y directions, find the expression for the total electric field E.

Hint:

$$\overrightarrow{E} = \mathrm{e_x}\, E_1 \sin(\omega t - \beta z) + \mathrm{e_y}\, E_2 \sin(\omega t - \beta z + \delta)$$

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Assignment Problems-II

5. From the given conditions in question 4, derive the equation

$$a\,E_x^{\,2} + b\,E_x E_y + cE_y^{\,2} = 1$$

where, $a = \dfrac{1}{E_1^2 \sin^2 \delta}$ $\quad b = \dfrac{2\cos\delta}{E_1 E_2 \sin^2 \delta}$ $\quad c = \dfrac{1}{E_2^2 \sin^2 \delta}$

Hint: $E_x = E_1 \sin(\omega t - \beta z)$ $\quad E_y = E_2 \sin(\omega t - \beta z + \delta)$

6. What do you understand by the axial ratio of a polarization ellipse?

7. What is a Poincaré Sphere and how is it related with the Polarization Ellipse?

8. Define all the four stokes parameters. Make a table and write the values of stokes parameters for (*i*) linearly polarized and (*ii*) circularly polarized waves.

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Assignment Problems-III

9. How many cross-correlations are formed between two antennas having two orthogonally polarized outputs?

10. Relate the cross-correlations of above problem with Stokes parameters when polarizations are (*i*) linear and (*ii*) circular.
Hint: See the final 8 equations of this lecture.

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THANK YOU