Chapter 2

Interferometry and Aperture Synthesis

A. P. Rao

2.1 Introduction

Radio astronomy is the study of the sky at radio wavelengths. While optical astronomy has been a field of study from time immemorial, the "new" astronomies viz. radioastronomy, X-ray, IR and UV astronomy are only about 50 years old. At many of these wavelengths it is essential to put the telescopes outside the confines of the Earth's atmosphere and so most of these "new" astronomies have become possible only with the advent of space technology. However, since the atmosphere is transparent in the radio band (which covers a frequency range of 10 MHz to 300 GHz or a wavelength range of approximately 1mm to 30m) radio astronomy can be done by ground based telescopes (see also Chapter 3).

The field of radioastronomy was started in 1923 when Karl Jansky, (working at the Bell Labs on trying to reduce the noise in radio receivers), discovered that his antenna was receiving radiation from outside the Earth's atmosphere. He noticed that this radiation appeared at the same sidereal (as opposed to solar) time on different days and that its source must hence lie far outside the solar system. Further observations enabled him to identify this radio source as the centre of the Galaxy. To honour this discovery, the unit of flux density in radioastronomy is named after Jansky where

$$1 \text{ Jansky} = 10^{-26} W m^{-2} Hz^{-1} \tag{2.1.1}$$

Radio astronomy matured during the second world war when many scientists worked on projects related to radar technology. One of the major discoveries of that period (made while trying to identify the locations of jamming radar signals), was that the sun is a strong emitter of radio waves and its emission is time variable. After the war, the scientists involved in these projects returned to academic pursuits and used surplus equipment from the war to rapidly develop this new field of radioastronomy. In the early phases, radioastronomy was dominated by radio and electronic engineers and the astronomy community, (dominated by optical astronomers), needed considerable persuasion to be convinced that these new radio astronomical discoveries were of relevance to astronomy in general. While the situation has changed considerably since then much of the jargon of radio astronomy (which is largely borrowed from electrical engineering) remains unfamiliar to a person with a pure physics background. The coherent detection techniques pioneered by radio astronomers also remains by and large not well understood by astronomers working at other wavelength bands. This set of lecture notes aims to familiarize students of physics (or students of astronomy at other wavelengths) with the techniques of radio astronomy.

2.2 The Radio Sky

The sky looks dramatically different at different wave bands and this is the primary reason multi-wavelength astronomy is interesting. In the optical band, the dominant emitters are stars, luminous clouds of gas, and galaxies all of which are thermal sources with temperatures in the range $10^3 - 10^4$ K. At these temperatures the emitted spectrum peaks in the optical band. Sources with temperatures outside this range and emitters of non thermal radiation are relatively weak emitters in the optical band but can be strong emitters in other bands. For example, cold (~ 100 K) objects emit strongly in the infra red and very hot objects ($> 10^5$ K) emit strongly in X-rays. Since the universe contains all of these objects one needs to make multiband studies in order to fully understand it.

For a thermal source with temperature greater than 100 K, the flux density in the radio band can be well approximated by the Rayleigh-Jeans Law[1], viz.

$$S = (2kT/\lambda^2)d\Omega \tag{2.2.2}$$

The predicted flux densities at radio wavelengths are miniscule and one might hence imagine that the radio sky should be dark and empty. However, radio observations reveal a variety of radio sources all of which have flux densities much greater than given by the Rayleigh-Jeans Law, i.e. the radio emission that they emit is not thermal in nature. Today it is known that the bulk of radio emission is produced via the synchrotron mechanism. Energetic electrons spiraling in magnetic fields emit synchrotron radiation. Unlike thermal emission where the flux density increases with frequency, for synchrotron emitters, the flux density increases with wavelength (see Figure 2.1). Synchrotron emitting sources are hence best studied at low radio frequencies.

The dominant sources seen in the radio sky are the Sun, supernova remnants, radio galaxies, pulsars etc. The Sun has a typical flux density of 10^5 Jy while the next strongest sources are the radio galaxy Cygnus A and the supernova remnant Cassiopeia A, both of which have flux densities of $\sim 10^4$ Jy. Current technology permits the detection of sources as weak as a few μJy. It turns out also that not all thermal sources are too weak to detect, the thermal emission from large and relatively nearby HII regions can also be detected easily in the radio band.

Radio emission from synchrotron and thermal emitters is “broad band”, i.e. the emission varies smoothly (often by a power law) over the whole radio band. Since the spectrum is relatively smooth, one can determine it by measurements of flux density at a finite number of frequencies. This is a major advantage since radio telescopes tend to be narrow band devices with small frequency spreads ($\Delta\nu/\nu \sim 0.1$). This is partly because it is not practical to build a single radio telescope that can cover the whole radio-band (see eg. Chapter 3) but mainly because radio astronomers share the radio band with a variety of other users (eg. radar, cellular phones, pagers, TV etc.) all of who radiate at power levels high enough to completely swamp the typical radio telescope. By international agreement, the radio spectrum is allocated to different users. Radio astronomy has

[1] The Rayleigh-Jeans Law, as can be easily verified, is the limit of the Plank law when $h\nu << kT$. This inequality is easily satisfied in the radio regime for generally encountered astrophysical temperatures.

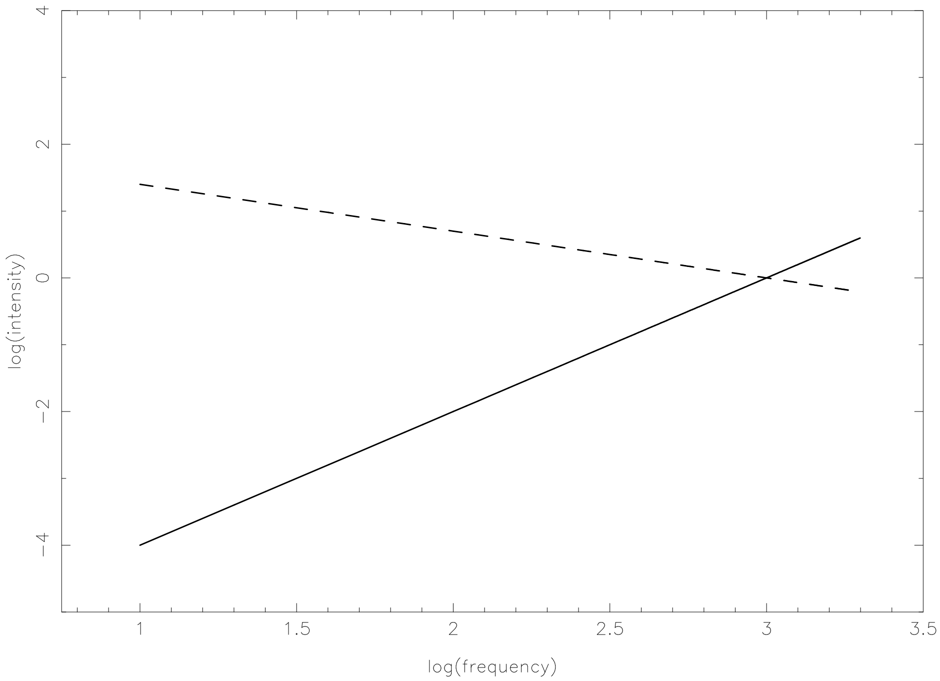

Figure 2.1: Intensity as a function of frequency ("power spectra") for synchrotron (dashed) and thermal (solid) radio sources.

a limited number of protected bands where no one else is permitted to radiate and most radio telescopes work only at these protected frequencies.

Several atoms and molecules have spectral lines in the radio band. For example, the hyperfine transition of the Hydrogen atom corresponds to a line with a wavelength of $\sim$ 21cm. Since atomic hydrogen (HI) is an extremely abundant species in the universe this line is one of the brightest naturally occurring radio lines. The HI 21cm line has been extensively used to study the kinematics of nearby galaxies. High quantum number recombination lines emitted by hydrogen and carbon also fall in the radio band and can be used to study the physical conditions in the ionized interstellar medium. Further the radio line emission from molecules like OH, SiO, H_2O etc. tend to be maser amplified in the interstellar medium and can often be detected to very large distances. Of course, these lines can be studied only if they fall within the protected radio bands. In fact, the presence of radio lines is one of the justifications for asking for protection in a specific part of the radio spectrum. While many of the important radio lines have been protected there are many outside the protected bands that cannot be studied, which is a source of concern. Further, with radio telescopes becoming more and more sensitive, it is possible to study lines like the 21cm line to greater and greater distances. Since in the expanding universe, distance translates to a redshift, this often means that these lines emitted by distant objects move out of the protected radio band and can become unobservable because of interference.

2.3 Signals in Radio Astronomy

A fundamental property of the radio waves emitted by cosmic sources is that they are stochastic in nature, i.e. the electric field at Earth due to a distant cosmic source can be treated as a random process[2]. Random processes can be simply understood as a generalization of random variables. Recall that a random variable x can be defined as follows. For every outcome o of some given experiment (say the tossing of a die) one assigns a given number to x. Given the probabilities of the different outcomes of the experiment one can then compute the mean value of x, the variance of x etc. If for every outcome of the experiment instead of a number one assigns a given function to x, then the associated process $x(t)$ is called a random process. For a fixed value of t, $x(t)$ is simply a random variable and one can compute its mean, variance etc. as before.

A commonly used statistic for random processes is the auto-correlation function. The auto-correlation function is defined as

$$r_{xx}(t,\tau) = \big\langle x(t)x(t+\tau)\big\rangle$$

where the angular brackets indicate taking the mean value. For a particularly important class of random processes, called wide sense stationary (WSS) processes the auto-correlation function is independent of changes of the origin of t and is a function of τ alone, i.e.

$$r_{xx}(\tau) = \big\langle x(t)x(t+\tau)\big\rangle$$

For $\tau = 0$, $r(\tau)$ is simply the variance σ^2 of $x(t)$ (which for a WSS process is independent of t).

The Fourier transform $S(\nu)$ of the auto-correlation function is called the power spectrum, i.e.

$$S(\nu) = \int_{-\infty}^{\infty} r_{xx}(\tau)e^{-i2\pi\tau\nu}d\tau$$

Equivalently, $S(\nu)$ is the inverse Fourier transform of $r(\tau)$ or

$$r_{xx}(\tau) = \int_{-\infty}^{\infty} S(\nu)e^{i2\pi\tau\nu}d\nu$$

Hence

$$r_{xx}(0) = \sigma^2 = \int_{-\infty}^{\infty} S(\nu)d\nu$$

i.e. since σ^2 is the "power" in the signal, $S(\nu)$ is a function describing how that power is distributed in frequency space, i.e. the "power spectrum".

A process whose auto-correlation function is a delta function has a power spectrum that is flat – such a process is called "white noise". As mentioned in Section 2.2, many radio astronomical signals have spectra that are relatively flat; these signals can hence be approximated as white noise. Radio astronomical receivers however have limited bandwidths, that means that even if the signal input to the receiver is white noise, the signal after passing through the receiver has power only in a finite frequency range. Its auto-correlation function is hence no longer a delta function, but is a sinc function (see Section 2.5) with a width $\sim 1/\Delta\nu$, where $\Delta\nu$ is the bandwidth of the receiver. The width of the auto-correlation function is also called the "coherence time" of the signal. The bandwidth $\Delta\nu$ is typically much smaller than the central frequency ν at which the radio receiver operates. Such signals are hence also often called "quasi-monochromatic" signals. Much like a monochromatic signal can be represented by a constant complex phasor, quasi-monochromatic signals can be represented by complex random processes.

Given two random processes $x(t)$ and $y(t)$, one can define a cross-correlation function

$$r_{xy}(\tau) = \big\langle x(t)y(t-\tau)\big\rangle$$

[2]see Chapter 1 for a more detailed discussion of topics discussed in this section.

where one has assumed that the signals are WSS so that the cross-correlation function is a function of τ alone. The cross-correlation function and its Fourier transform, the cross power spectrum, are also widely used in radio astronomy.

We have so far been dealing with random processes that are a function of time alone. The signal received from a distant cosmic source is in general a function both of the receivers location as well as of time. Much as we defined temporal correlation functions above, one can also define spatial correlation functions. If the signal at the observer's plane at any instant is E(**r**), then spatial correlation function is defined as:

$$V(\mathbf{x}) = \left\langle E(\mathbf{r})E^*(\mathbf{r}+\mathbf{x})\right\rangle$$

Note that strictly speaking the angular brackets imply ensemble averaging. In practice one averages over time[3] and assumes that the two averaging procedures are equivalent. The function V is referred to as the "visibility function" (or just the "visibility") and as we shall see below, it is of fundamental interest in interferometry.

2.4 Interferometry

2.4.1 The Need for Interferometry

The idea that the resolution of optical instruments is limited due to the wave nature of light is familiar to students of optics and is embodied in the Rayleigh's criterion which states that the angular resolution of a telescope/microscope is ultimately diffraction limited and is given by

$$\theta \sim \lambda/D \tag{2.4.3}$$

where D is some measure of the aperture size. The need for higher angular resolution has led to the development of instruments with larger size and which operate at smaller wavelengths. In radioastronomy, the wavelengths are so large that even though the sizes of radio telescopes are large, the angular resolution is still poor compared to optical instruments. Thus while the human eye has a diffraction limit of $\sim\ 20''$ and even modest optical telescopes have diffraction limits[4] of $0.1''$, even the largest radio telescopes (300m in diameter) have angular resolutions of only $\sim 10'$ at 1 metre wavelength. To achieve higher resolutions one has to either increase the diameter of the telescope further (which is not practical) or decrease the observing wavelength. The second option has led to a tendency for radio telescopes to operate at centimetre and millimetre wavelengths, which leads to high angular resolutions. These telescopes are however restricted to studying sources that are bright at cm and mm wavelengths. To achieve high angular resolutions at metre wavelengths one need telescopes with apertures that are hundreds of kilometers in size. Single telescopes of this size are clearly impossible to build. Instead radio astronomers achieve such angular resolutions using a technique called aperture synthesis. Aperture synthesis is based on interferometry, the principles of which are familiar to most physics students. There is in fact a deep analogy between the double slit experiment with quasi-monochromatic light and the radio two element interferometer. Instead of setting up this analogy we choose the more common route to radio interferometry via the van Cittert-Zernike theorem.

[3] For typical radio receiver bandwidths of a few MHz, the coherence time is of the order of micro seconds, so in a few seconds time one gets several million independent samples to average over.

[4] The actual resolution achieved by these telescopes is however usually limited by atmospheric seeing.

2.4.2 The Van Cittert Zernike Theorem

The van Cittert-Zernike theorem relates the spatial coherence function $V(\mathbf{r}_1, \mathbf{r}_2) = \left\langle \mathrm{E}(\mathrm{r}_1) E^*(\mathrm{r}_2) \right\rangle$ to the distribution of intensity of the incoming radiation, $\mathcal{I}(\mathbf{s})$. It shows that the spatial correlation function $V(\mathbf{r}_1, \mathbf{r}_2)$ depends only on $\mathbf{r}_1 - \mathbf{r}_2$ and that if all the measurements are in a plane, then

$$V(\mathbf{r}_1, \mathbf{r}_2) = \mathcal{F}\{I(\mathbf{s})\} \tag{2.4.4}$$

where $\mathcal{F}$ implies taking the Fourier transform. Proof of the van Cittert-Zernike theorem can be found in a number of textbooks, eg. “Optics” by Born and Wolf, “Statistical Optics” by Goodman, “Interferometry and Synthesis in radio astronomy” by Thompson et al. We give here only a rough proof to illustrate the basic ideas.

Let us assume that the source is distant and can be approximated as a brightness distribution on the celestial sphere of radius R (see Figure 2.2). Let the electric field[5] at a point $P_1'(x_1', y_1', z_1')$ at the source be given by $\mathcal{E}(P_1')$. The field $E(P_1)$ at the observation point $P_1(x_1, y_1, z_1)$ is given by[6]

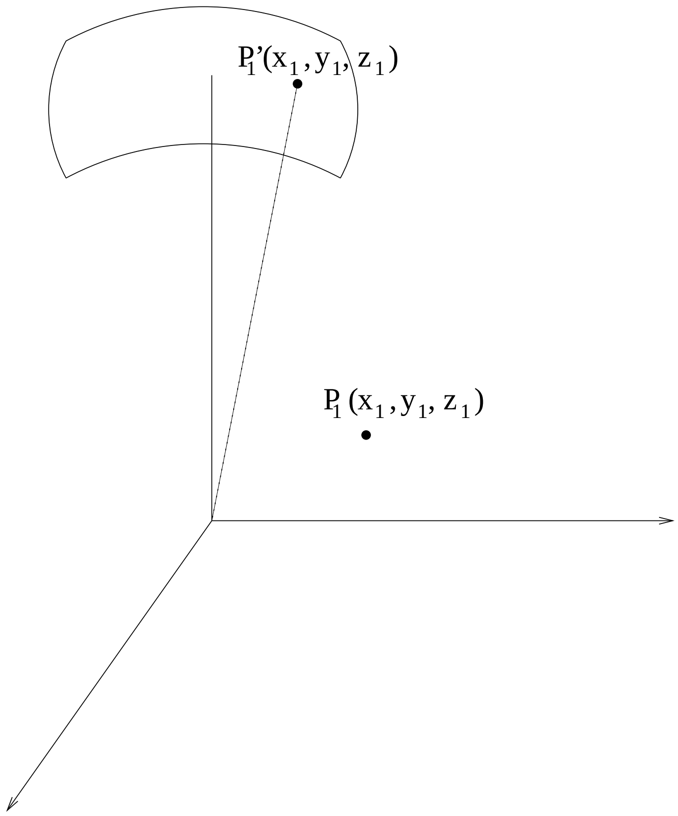

Figure 2.2: Geometry for the van Cittert-Zernike theorem

$$E(P_1) = \int \mathcal{E}(P_1') \frac{e^{-ikD(P_1', P_1)}}{D(P_1', P_1)} d\Omega_1 \tag{2.4.5}$$

[5]We assume here for the moment that the electric field is a scalar quantity. See Chapter 15 for the extension to vector fields.

[6]Where we have invoked Huygens principle. A more rigorous proof would use scalar diffraction theory.

where $D(P_1', P_1)$ is the distance between P_1' and P_1. Similarly if $E(P_2)$ is the field at some other observing point $P_2(x_2, y_2, z_2)$ then the cross-correlation between these two fields is given by

$$\left\langle E(P_1)E^*(P_2)\right\rangle = \int \left\langle \mathcal{E}(P_1')\mathcal{E}^*(P_2')\right\rangle \frac{e^{-ik[D(P_1',P_1)-D(P_2',P_2)]}}{D(P_1',P_1)D(P_2',P_2)} d\Omega_1 d\Omega_2 \tag{2.4.6}$$

If we further assume that the emission from the source is spatially incoherent, i.e. that $\left\langle \mathcal{E}(P_1')\mathcal{E}^*(P_2')\right\rangle = 0$ except when $P_1' = P_2'$, then we have

$$\left\langle E(P_1)E^*(P_2)\right\rangle = \int \mathcal{I}(P_1') \frac{e^{-ik[D(P_1',P_1)-D(P_1',P_2)]}}{D(P_1',P_1)D(P_1',P_2)} d\Omega_1 \tag{2.4.7}$$

where $\mathcal{I}(P_1')$ is the intensity at the point P_1'. Since we have assumed that the source can be approximated as lying on a celestial sphere of radius R we have $x_1' = R\cos(\theta_x) = Rl$, $y_1' = R\cos(\theta_y) = Rm$, and $z_1' = R\cos(\theta_z) = Rn$; (l, m, n) are called "direction cosines". It can be easily shown[7] that $l^2 + m^2 + n^2 = 1$ and that $d\Omega = \frac{dl\ dm}{\sqrt{1-l^2-m^2}}$. We then have:

$$\begin{aligned}
D(P_1', P_1) &= \left[(x_1' - x_1)^2 + (y_1' - y_1)^2 + (z_1' - z_1)^2\right]^{1/2} && (2.4.8)\\
&= \left[(Rl - x_1)^2 + (Rm - y_1)^2 + (Rn - z_1)^2\right]^{1/2} && (2.4.9)\\
&= R\left[(l - x_1/R)^2 + (m - y_1/R)^2 + (n - z_1/R)^2\right]^{1/2} && (2.4.10)\\
&\simeq R\left[(l^2 + m^2 + n^2) - 2/R(lx_1 + my_1 + nz_1)\right]^{1/2} && (2.4.11)\\
&\simeq R - (lx_1 + my_1 + nz_1) && (2.4.12)\\
& && (2.4.13)
\end{aligned}$$

Putting this back into equation 2.4.7 we get

$$\left\langle E(P_1)E^*(P_2)\right\rangle = \frac{1}{R^2}\int \mathcal{I}(l, m) e^{-ik[l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)]} \frac{dl\ dm}{\sqrt{1 - l^2 - m^2}} \tag{2.4.14}$$

Note that since $l^2 + m^2 + n^2 = 1$, the two directions cosines (l, m) are sufficient to uniquely specify any given point on the celestial sphere, which is why the intensity $\mathcal{I}$ has been written out as a function of (l, m) only. It is customary to measure distances in the observing plane in units of the wavelength λ, and to define "baseline co-ordinates" u, v, w such that $u = (x_2 - x_1)/\lambda$, $v = (y_2 - y_1)/\lambda$, and $w = (z_2 - z_1)/\lambda$. The spatial correlation function $\left\langle E(P_1)E^*(P_2)\right\rangle$ is also referred to as the "visibility" $\mathcal{V}(u, v, w)$. Apart from the constant factor $1/R^2$ (which we will ignore hence forth) equation 2.4.14 can then be written as

$$\mathcal{V}(u, v, w) = \int \mathcal{I}(l, m) e^{-i2\pi[lu+mv+nw]} \frac{dl\ dm}{\sqrt{1 - l^2 - m^2}} \tag{2.4.15}$$

This fundamental relationship between the visibility and the source intensity distribution is the basis of radio interferometry. In the optical literature this relationship is also referred to as the van Cittert-Zernike theorum.

Equation 2.4.15 resembles a Fourier transform. There are two situations in which it does reduce to a Fourier transform. The first is when the observations are confined to a the $U - V$ plane, i.e. when $w = 0$. In this case we have

$$\mathcal{V}(u, v) = \int \frac{\mathcal{I}(l, m)}{\sqrt{1 - l^2 - m^2}} e^{-i2\pi[lu+mv]} dl\ dm \tag{2.4.16}$$

[7] see for example, Christiansen & Hogbom, "Radio telescopes", Cambridge University Press

i.e. the visibility $\mathcal{V}(u, v)$ is the Fourier transform of the modified brightness distribution $\frac{\mathcal{I}(l,m)}{\sqrt{1-l^2-m^2}}$. The second situation is when the source brightness distribution is limited to a small region of the sky. This is a good approximation for arrays of parabolic antennas because each antenna responds only to sources which lie within its primary beam (see Chapter 3). The primary beam is typically $< 1^o$, which is a very small area of sky. In this case $n = \sqrt{1 - l^2 - m^2} \simeq 1$. Equation 2.4.15 then becomes

$$\mathcal{V}(u, v, w) = e^{-i2\pi w} \int \mathcal{I}(l, m) e^{-i2\pi[lu+mv]} dl\ dm \tag{2.4.17}$$

or if we define a modified visibility $\widetilde{\mathcal{V}}(u, v) = \mathcal{V}(u, v, w)e^{i2\pi w}$ we have

$$\widetilde{\mathcal{V}}(u, v) = \int \mathcal{I}(l, m) e^{-i2\pi[lu+mv]} dl\ dm \tag{2.4.18}$$

2.4.3 Aperture Synthesis

As we saw in the previous section, the spatial correlation of the electric field in the U-V plane is related to the source brightness distribution. Further, for the typical radio array the relationship between the measured visibility and the source brightness distribution is a simple Fourier transform. Correlation of the voltages from any two radio antennas then allows the measurement of a single Fourier component of the source brightness distribution. Given sufficient number of measurements the source brightness distribution can then be obtained by Fourier inversion. The derived image of the sky is usually called a “map” in radio astronomy, and the process of producing the image from the visibilities is called “mapping”.

The radio sky (apart from a few rare sources) does not vary[8]. This means that it is not necessary to measure all the Fourier components simultaneously. Thus for example one can imagine measuring all required Fourier components with just two antennas, (one of which is mobile), by laboriously moving the second antenna from place to place. This method of gradually building up all the required Fourier components and using them to image the source is called “aperture synthesis”. If for example one has measured all Fourier components up to a baseline length of say 25 km, then one could obtain an image of the sky with the same resolution as that of a telescope of aperture size 25 km, i.e. one has synthesized a 25 km aperture. In practice one can use the fact that the Earth rotates to sample the U-V plane quite rapidly. As seen from a distant cosmic source, the baseline vector between two antennas on the Earth is continuously changing because of the Earth’s rotation (see Figure 2.3). Or equivalently, as the source rises and sets the Fourier components measured by a given pair of antennas is continuously changing. If one has an array of N antennas spread on the Earth’s surface, then at any given instant one measures ${}^NC_2$ Fourier components (or in radio astronomy jargon one has ${}^NC_2$ samples in the U-V plane). As the Earth rotates one samples more and more of the U-V plane. For arrays like the GMRT with 30 antennas, if one tracks a source from rise to set, the sampling of the U-V plane is sufficiently dense to allow extremely high fidelity reconstructions of even complex sources. This technique of using the Earth’s rotation to improve “U-V coverage” was traditionally called “Earth rotation aperture synthesis”, but in modern usage is usually also simply referred to as “aperture synthesis”.

From the inverse relationship of Fourier conjugate variables it follows that short baselines are sensitive to large angular structures in the source and that long baselines are

[8] Or, in the terminology of random processes cosmic radio signals are stationary, i.e. their statistical properties like the mean, auto and cross-correlation functions etc. are independent of the absolute time.

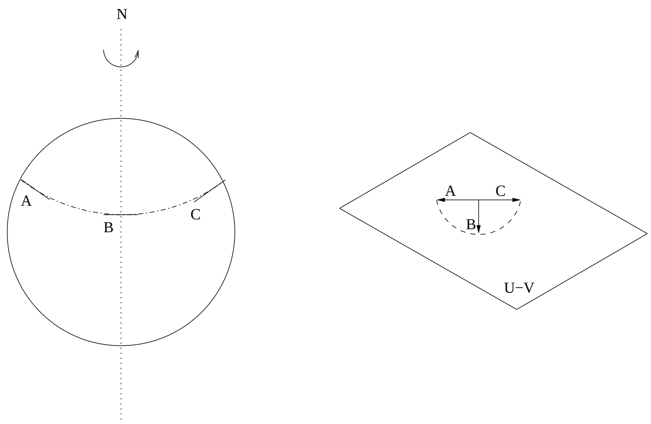

Figure 2.3: The track in the U-V plane traced out by an east-west baseline due to the Earth's rotation.

sensitive to fine scale structure. To image large, smooth sources one would hence like an array with the antennas closely packed together, while for a source with considerable fine scale structure one needs antennas spread out to large distances. The array configuration hence has a major influence on the kind of sources that can be imaged. The GMRT array configuration consists of a combination of a central 1x1 km cluster of densely packed antennas and three 14 km long arms along which the remaining antennas are spread out. This gives a combination of both short and long spacings, and gives considerable flexibility in the kind of sources that can be imaged. Arrays like the VLA on the other hand have all their antennas mounted on rails, allowing even more flexibility in determining how the U-V plane is sampled.

Other chapters in these notes discuss the practical details of aperture synthesis. Chapter 3 discusses how one can use radio antennas and receivers to measure the electric field from cosmic sources. For an N antenna array one needs to measure ${}^{N}C_2$ correlations simultaneously, this is done by a (usually digital) machine called the "correlator". The spatial correlation that one needs to measure (see equation 2.4.6) is the correlation between the instantaneous fields at points P_1 and P_2. In an interferometer the signals from antennas at points P_1 and P_2 are transported by cable to some central location where the correlator is – this means that the correlator has also to continuously adjust the delays of the signals from different antennas before correlating them. This and other corrections that need to be made are discussed in Chapter 4, and exactly how these corrections are implemented in the correlator are discussed in Chapters 8 and 9. The astronomical calibration of the measured visibilities is discussed in Chapter 5, while Chapter 16 deals with the various ways in which passage through the Earth's ionosphere corrupts the astronomical signal. Chapters 10, 12 and 14 discuss the nitty gritty of going from the calibrated visibilities to the image of the sky. Chapters 13 and 15 discuss two refinements, viz. measuring the spectra and polarization properties of the sources respectively.

2.5 The Fourier Transform

The Fourier transform $U(\nu)$ of a function $u(t)$ is defined as

$$U(\nu) = \int_{-\infty}^{\infty} u(t) e^{-i2\pi\nu t} dt$$

and can be shown to exist for any function $u(t)$ for which

$$\int_{-\infty}^{\infty} |u(t)| dt < \infty$$

The Fourier transform is invertible, i.e. given $U(\nu)$, $u(t)$ can be obtained using the inverse Fourier transform, viz.

$$u(t) = \int_{-\infty}^{\infty} U(\nu) e^{i2\pi\nu t} d\nu$$

Some important properties of the Fourier transform are listed below (where by convention capitalized functions refer to the Fourier transform)

1. Linearity
$$\mathcal{F}\{au(t) + bv(t)\} = aU(\nu) + bV(\nu)$$
where a, b are arbitrary complex constants.
2. Similarity
$$\mathcal{F}\{u(at)\} = \frac{1}{a} U(\frac{\nu}{a})$$
where a is an arbitrary real constant.
3. Shift
$$\mathcal{F}\{u(t-a)\} = e^{-i2\pi a} U(\nu)$$
where a is an arbitrary real constant.
4. Parseval's Theorem
$$\int_{-\infty}^{\infty} |u(t)|^2 dt = \int_{-\infty}^{\infty} |U(\nu)|^2 d\nu$$
5. Convolution Theorem
$$\mathcal{F} \int_{-\infty}^{\infty} u(t) v(t-\tau) dt = U(\nu) V(\nu)$$
6. Autocorrelation Theorem
$$\mathcal{F} \int_{-\infty}^{\infty} u(t) u(t+\tau) dt = |U(\nu)|^2$$

Some commonly used Fourier transform pairs are:

Table 2.1: Fourier transform pairs

Function	Transform
$e^{\pi t^2}$	$e^{\pi \nu^2}$
1	$\delta(\nu)$
$\cos(\pi t)$	$\frac{1}{2}\delta(\nu - \frac{1}{2}) + \frac{1}{2}\delta(\nu + \frac{1}{2})$
$\sin(\pi t)$	$\frac{i}{2}\delta(\nu - \frac{1}{2}) - \frac{i}{2}\delta(\nu + \frac{1}{2})$
$rect(t)$	$sinc(\nu)$